Continuous-Time Markov Networks: From Merging Rules to Synchronizing Control

Changxi Li, Lin Lin, Wenjun Xiong, James Lam, *Fellow, IEEE*, and Ka-Wai Kwok, *Senior Member, IEEE*

***Abstract*—This article characterizes the state-space representation and control Lyapunov theory for continuous-time Markov networks (CT-MNs), which constitute a family of Markov chains endowed with network structures, serving as the theoretical basis for the investigations of several swarm behaviors. First, the notions of coupling Markov property and conditional independent property are introduced, enabling a CT-MN to be merged as a decentralized continuous-time Markov chain (CT-MC). Then, explicit formulas for the transition rate matrix of the augmented CT-MCs are derived. Consequently, key problems in CT-MNs can be reformulated as set-stability or set-stabilization problems for the augmented CT-MCs, with feedback controllers designed via linear programming subject to Lyapunov-based stability constraints. Furthermore, leader-follower synchronization and output regulation of CT-MNs are addressed. Finally, numerical examples validate the theoretical results and demonstrate their effectiveness in practical applications.**

***Index Terms*—Continuous-time Markov networks (CT-MNs), control Lyapunov functions, Markov chains, leader-follower synchronization, output regulation, state–space representation.**

Received 5 March 2025; accepted 31 May 2025. Date of publication 17 June 2025; date of current version 5 December 2025. This work was supported in part by the National Natural Science Foundation of China under Grant 62103232, Grant 62373308, and Grant 62273286; in part by the Natural Science Foundation of Shandong Province under Grant ZR2021QF005; in part by Qilu Young Scholars Project of Shandong University; in part by Taishan Scholar Project of Shandong Province under Grant tsqn202312033; in part by the Central Leading Local Science and Technology Development Fund under Grant 2023ZYD0009; in part by the General Research Fund (GRF) under Grant 17200124; in part by Guangdong Basic Research and Applied Research Fund 2024A1515011509; in part by the Research Grants Council of Hong Kong under Grant STG1/E-401/23-N, Grant C4026-21G, and Grant 17209021; in part by the Center for Transformative Garment Production and the Multi-scale Medical Robotics Center under the InnoHK Initiative of the Innovation and Technology Commission of the HKSAR Government; and in part by the Fundamental Research Funds for the Central Universities. Recommended by Associate Editor M. Zamani.
(Corresponding author: Lin Lin.)

Changxi Li is with the School of Mathematics, Shandong University, Jinan 250100, China, and also with the Key Laboratory of Smart Manufacturing in Energy Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai 200237, China (e-mail: lichangxi@sdu.edu.cn).

Lin Lin and James Lam are with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong (e-mail: linlin00wa@gmail.com; james.lam@hku.hk).

Wenjun Xiong is with the School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 610056, China (e-mail: xwenjun2@gmail.com).

Ka-Wai Kwok is with the Department of Mechanical Engineering, University of Hong Kong, Hong Kong, and also with the Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Hong Kong (e-mail: kwokkw@mae.cuhk.edu.hk).

Digital Object Identifier 10.1109/TAC.2025.3580684

I. Introduction

IN RECENT years, on account of the capability of characterizing the prevalent engineering systems and complex networks, Markov chains have attracted widespread research interest [1]. In particular, *homogeneous* Markov chains are generally used for describing system dynamics over finite-cardinality state spaces, where transition probability matrices usually describe the transitions among state pairs. Illustrative engineering examples include the stochastic networks over finite fields [2]. Boarder speaking, as long as the dynamics of systems evolve along with a finite set of candidates, inherited states are accordingly selected in certain probability rules, and therefore, certain potential Markov chains exist to equivalently describe the whole system behavior. Such circumstances commonly exist in gene regulatory networks [3], [4], [5], robotics synthesis [6], game theory [7], [8], and so on. The investigations on Markov chains are also highly pertinent to machine learning, for which hidden Markov chains are always utilized to match the time-series data [9] with the target of learning the optimal decision strategies [10].

Typically, due to the distinct evolutionary nature of Markov chains, they can be classified into discrete-time Markov chains (DT-MCs) and continuous-time Markov chains (CT-MCs). Currently, the predominant focus of research has been on DT-MCs, largely because they provide advantages in characterizing macroscopically imprecise information. In this area, previous works have primarily concentrated on the steady-state analysis of DT-MCs. For example, Zhao and Cheng [11] characterized absorbing DT-MCs under stochastic conditions by establishing a necessary and sufficient condition involving the existence of a rooted in-tree in their underlying state transition graphs. Subsequent work [12] examined the asymptotic convergence of DT-MCs in terms of mathematical expectation and developed a series of linear programming constraints to generate Lyapunov functions. Later, Guo et al. [13] extended the mathematical concept of asymptotic convergence for CT-MCs to asymptotic convergence in distribution, clarifying the transition from one-point absorbing states to set absorbing states. In addition, a state feedback scheme was proposed in [14] to ensure that controlled DT-MCs globally absorb toward a preassigned state. Beyond asymptotic convergence, the finite-time convergence of DT-MCs was shown in [15] to be equivalent to the existence of a topological sorting of their state transition graph. Furthermore, research has also addressed the observability of DT-MCs; for instance, recent work [16] investigated observable and detectable graphs to characterize the observability and detectability of DT-MCs from a graph-theoretic perspective.

While DT-MCs have been extensively studied in the literature, the continuous-time counterpart remains relatively underexplored [17]. Unlike DT-MCs, CT-MCs utilize transition rate matrices instead of transition probability matrices as their constant parameters [17], [18], [19], [20], [21]. This distinction requires the development of new theoretical frameworks specifically tailored for CT-MCs. However, current research on CT-MCs remains largely confined to decentralized chain structures, leaving their potential applications in network science insufficiently explored. To bridge this gap, we propose studying CT-MCs integrated with network graphs to capture the coupling relationships between interconnected subchains. Such models are termed as continuous-time Markov networks (CT-MNs) in this article. A key motivation for studying CT-MNs lies in their wide-ranging applications, particularly in synchronization problems that underlie critical tasks such as tracking control [22], consensus realization [2], and output regulation [23]. Therefore, the investigation of CT-MNs presents both important theoretical contributions and practical benefits beyond what can be achieved with conventional CT-MC analysis.

To date, the analysis of swarm behaviors in CT-MNs remains an open challenge that cannot be resolved through direct extensions of existing results for either CT-MCs or discrete-time Markov networks (DT-MNs). While synchronization problems in DT-MNs can typically be addressed by constructing augmented state variables that aggregate network states [24], this approach proves inadequate for CT-MCs due to insufficient characterization of their coupling structures [17]. The fundamental difficulty arises when attempting to merge state vectors within the stochastic expectation form under general conditions. Crucially, the fusing rules established for DT-MNs [24] rely heavily on the power-reduced property of matrices [25], which falls to hold for arbitrary noncanonical vectors. These limitations present significant theoretical obstacles in CT-MN analysis, particularly in determining the precise conditions, under which traditional power-reduced properties can be extended to probabilistic state vectors.

Based on the aforementioned discussions, this article establishes fundamental conditions for analyzing swarm behaviors in CT-MNs by introducing the coupling Markov property and the conditional independence property. These properties enable the amalgamation and separation of probability vectors in continuous-time settings. Building on this foundation, we develop a continuous-time state-space representation for CT-MNs to address critical challenges in variable augmentation, generalizing the traditional methodology in [25]. Furthermore, a Lyapunov-based control design scheme is developed for CT-MNs that guarantees asymptotic stability. In contrast to existing approaches [17], this scheme reduces the control synthesis problem to a linear programming problem constrained by the augmented transition rate matrix (TRM), yielding superior computational efficiency. In addition, this work provides complete solutions to the leader-follower synchronization and output regulation of CT-MNs by establishing the essential fusing conditions, which were absent in prior studies [19], [26].

The rest of this article is organized as follows. Section II introduces the mathematical models of CT-MNs and continuous-time control Markov networks (CT-CMNs), building upon the fundamental definitions of CT-MCs and continuous-time control Markov chains (CT-CMCs). This section also discusses the research motivation and problem statements. Section III develops the state-space representation framework for both CT-MNs and CT-CMNs. Based on this foundation, Section IV studies the stability and stabilization problems. These theoretical developments are then specialized in Section V to address leader-follower synchronization and output regulation problems. The practical relevance of our theoretical results is demonstrated in Section VI through three illustrative examples. Finally, Section VII concludes this article.

Notations: $\mathcal{R}^n$ is the Euclidean space of all real n-vectors. $\mathcal{R}_{m\times n}$ is the set of $m \times n$ real matrices. $\mathscr{D}_k := \{1, 2, \ldots, k\}$. $\mathbf{1}_m$ is an m-dimensional vector with identical entries, and I_n is an $n \times n$ identity matrix. For an $m \times n$ matrix $A \in \mathcal{R}_{m\times n}$, $[A]_{i,j}$ is the element in the ith row and jth column, $\mathrm{Row}_r(A)$ is the rth row of matrix A, and $\mathrm{Col}_r(A)$ is the rth column of matrix A. Besides, $\mathrm{Col}(A)$ signifies the set of columns of A, and $A^\top$ denotes the transpose of A. Let $\Delta_n := \mathrm{Col}(I_n)$ and $\delta_n^i := \mathrm{Col}_i(I_n)$. Denote by $\mathcal{L}_{s\times r}$ the set of all $s \times r$ canonical matrices whose columns belong to Δ_n. $\mathcal{P}^n$ is the set of all n-dimensional probability vectors and $\mathcal{P}_{m\times n}$ is the set of all $m \times n$ stochastic matrices. The Khatri–Rao product of matrix $A \in \mathcal{R}_{p\times n}$ and matrix $B \in \mathcal{R}_{q\times n}$ is defined as [25]: $\mathrm{Col}_i(A * B) = \mathrm{Col}_i(A) \otimes \mathrm{Col}_i(B)$, $i \in \mathcal{D}_n$, where $\otimes$ is the Kronecker product. Moreover, the Kronecker sum $\oplus$ of matrix $A \in \mathcal{R}_{n\times n}$ and $B \in \mathcal{R}_{m\times m}$ is defined as $A \oplus B = A \otimes I_m + I_n \otimes B$.

II. Mathematical Models and Problem Formulation

This section introduces the standard mathematical models for CT-MNs and their controlled counterparts, building upon the conventional CT-MC theory. As traditional approaches prove inadequate for CT-MNs and CT-CMNs, we formulate the key problems addressed in this work.

A. Continuous-Time Markov Chains (CT-MCs)

According to [17] and [27], CT-MCs are defined as continuous-time homogenous Markov processes $x(t)$ over a finite state space $\mathscr{D}_k$, governed by the transition probability matrix (TPM) $P(t) \in \mathcal{P}_{k\times k}$:

$$[P(t)]_{i,j} := \Pr\{x(\theta + t) = i \,|\, x(\theta) = j\},\ i, j \in \mathscr{D}_k \quad (1)$$

with initial condition $P(0) = I_k$. The homogeneousness property implies time invariance

$$P(t + \theta) = P(t)P(\theta), \forall t, \theta \geq 0.$$

Assuming that each parameter in matrix $P(t)$ is continuous and differentiable, we have

$$\lim_{t\to 0^+} P(t) = P(0) = I_k.$$

It yields the TRM

$$Q := \lim_{t\to 0^+} \dot{P}(t) \in \mathcal{R}_{k\times k}. \quad (2)$$

Here, TRM Q, also known as the intensity matrix and infinitesimal generator matrix [28], characterizes instantaneous transition rates, where $[Q]_{i,j} \geq 0$ represents the transition rate from state j to i. Moreover, TPM $P(t)$ satisfies

$$\dot{P}(t) = QP(t) \quad (3)$$

whose solution is $P(t) = e^{Qt}$. The TRM Q exhibits the following fundamental properties:

1) *Negative Diagonal:* $[Q]_{i,i} \leq 0$ for all $i \in \mathscr{D}_k$;
2) *Nonnegative Off-Diagonals:* $[Q]_{i,j} \geq 0$ for all $i, j \in \mathscr{D}_k$ with $i \neq j$;
3) *Conservation Condition:* $\sum_{i\in\mathscr{D}_k}[Q]_{i,j} = 0$ for all $j \in \mathscr{D}_k$.

Unlike DT-MCs with a constant TPM, CT-MCs are fundamentally characterized by their TRM $Q \in \mathcal{R}_{k\times k}$. The probability distribution vector (PDV) $\mathbf{p}(t) \in \mathcal{P}^k$ of state $x(t)$, defined as $[\mathbf{p}(t)]_i = \Pr\{x(t) = i\}$, evolves according to

$$\begin{cases}\dot{\mathbf{p}}(t) = Q\mathbf{p}(t)\\ \mathbf{p}(0) = \mathbf{p}_0 \in \mathcal{P}^k\end{cases} \tag{4}$$

with solution $\mathbf{p}(t; \mathbf{p}_0) = e^{Qt}\mathbf{p}_0$. We hereafter denote a CT-MC with TRM Q as CT-MC $\{Q\}$.

B. Continuous-Time Control Markov Chains (CT-CMCs)

While CT-MCs evolve autonomously, CT-CMCs incorporate both state dynamics $x(t) \in \mathscr{D}_k$ and control inputs $u(t) \in \mathscr{D}_q$, where k and q denote the state and control space cardinalities, respectively.

In particular, CT-CMCs can be interpreted as switched CT-MCs with q operational modes, where control input u serves as the switching signal [18]. When $u(\theta) = r$ at time $\theta \geq 0$, the system activates mode r, characterized by its TPM $P_r(t) \in \mathcal{P}_{k\times k}$:

$$[P_r(t)]_{i,j} = \Pr\{x(\theta+t) = i | x(\theta) = j, u(\theta) = r\}, \forall i, j \in \mathscr{D}_k.$$

Likewise to TPM $P(t)$, TPM $P_r(t)$ of each mode $r \in \mathscr{D}_q$ satisfies the properties:

1) $P_r(0) = I_k$
2) $\lim_{t\to 0^+} P_r(t) = I_k$
3) entries are continuous and differential subject to time t.

The TRM for mode r is given by $Q_r^c := \lim_{t\to 0^+} \dot{P}_r(t)$, which satisfies the Kolmogorov equation as

$$\dot{P}_r(t) = Q_r^c P_r(t),\ r \in \mathscr{D}_q \tag{5}$$

with solution $P_r(t) = e^{Q_r^c t}$.

The complete CT-CMC system is characterized by the composite TRM

$$Q^c := \left[Q_1^c, Q_2^c, \ldots, Q_q^c\right] \in \mathcal{R}_{k\times qk}.$$

The PDV $\mathbf{p}(t)$ evolves according to

$$\begin{cases}\dot{\mathbf{p}}(t) = Q^c\left(\vec{u}(t) \otimes I_k\right)\mathbf{p}(t)\\ \mathbf{p}(0) = \mathbf{p}_0 \in \mathcal{P}^k\end{cases} \tag{6}$$

where $\vec{u}(t) := \delta_q^{u(t)}$ is the canonical basis vector representation of the control input $u(t)$. Hereafter, we denote such systems as CT-CMC $\{Q^c\}$.

C. Continuous-Time Markov Networks (CT-MNs)

Recently, single-structured CT-MCs have been investigated in [17] and [19]. However, gaps still exist between single chains and distributed interconnected chains. Unlike CT-MCs, CT-MNs are structurally connected and multicomponent. The concept of CT-MNs was first proposed in [29] and [30].

A CT-MN is characterized by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{1, 2, \ldots, n\}$ is the component set, and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ defines interconnection topology. For each component $i \in \mathcal{N}$, subset $\mathcal{N}_i := \{j \mid (i, j) \in \mathcal{E}\}$ collects the in-neighbors of component i, and variable $x_i \in \mathscr{D}_{k_i}$ represents its state that has k_i possible values. The global network state combines all components

$$x(t) = \left(x_1^\top(t), x_2^\top(t), \ldots, x_n^\top(t)\right)^\top \in \prod_{j\in\mathcal{N}} \mathscr{D}_{k_j}.$$

Note that each component $i \in \mathcal{N}$ is governed by the TPM $P_i(t) \in \mathcal{P}_{k_i\times k_i k_{\mathcal{N}_i}}$, where $k_{\mathcal{N}_i} := \prod_{j\in\mathcal{N}_i} k_j$ is the cardinality of the neighbor state set.

In this article, the considered CT-MNs satisfy two fundamental properties.

1) *Coupling Markov Property:* For each component $i \in \mathcal{N}$ with its neighbor state vector as

$$x_{\mathcal{N}_i} := \left(x_{j_1}^\top, x_{j_2}^\top, \ldots, x_{j_{|\mathcal{N}_i|}}^\top\right)^\top \in \prod_{j\in\mathcal{N}_i} \mathscr{D}_{k_j},$$

the transition probability of $x_i \in \mathscr{D}_{k_i}$ depends on $x_{\mathcal{N}_i} \in \prod_{j\in\mathcal{N}_i} \mathscr{D}_{k_j}$ through TPM $[P_i(t)]^\gamma \in \mathcal{P}_{k_i\times k_i}$, defined as

$$\begin{aligned}&[P_i(t)]^\gamma_{\alpha,\beta}\\ &:= \Pr\left\{x_i(t+\theta) = \alpha \,|\, x_i(\theta) = \beta, x_{\mathcal{N}_i}(\theta) = \gamma\right\}\end{aligned} \tag{7}$$

for $\alpha, \beta \in \mathscr{D}_{k_i}$ and $\gamma \in \mathscr{D}_{k_{\mathcal{N}_i}}$. Each $\gamma \in \mathscr{D}_{k_{\mathcal{N}_i}}$ represents a state configuration in $\prod_{j\in\mathcal{N}_i} \mathscr{D}_{k_j}$ and induces a TPM as $[P_i(t)]^\gamma$ for the CT-MC.

2) *Conditional Independence Property:* Component states $x_1(t+\theta), x_2(t+\theta), \ldots, x_n(t+\theta)$ for $t > 0$ and all $\theta \geq 0$, evolve conditionally independent on the current state $x(\theta)$ as follows:

$$\Pr\{x(t+\theta) \mid x(\theta)\} = \prod_{i=1}^{n} \Pr\{x_i(t+\theta) \mid x(\theta)\}. \tag{8}$$

Each component $i \in \mathcal{N}$ of a CT-MN, characterized by TPM $P_i(t)$, can be interpreted as a switched CT-MC with $k_{\mathcal{N}_i}$ modes, where neighbor state $x_{\mathcal{N}_i}$ serves as the switching signal. The TRM $[Q_i]^\gamma \in \mathcal{R}_{k_i\times k_i}$ and TPM $[P_i(t)]^\gamma \in \mathcal{P}_{k_i\times k_i}$ for each mode $\gamma \in \mathscr{D}_{k_{\mathcal{N}_i}}$ satisfy

$$[Q_i]^\gamma = \lim_{t\to 0^+}\left[\dot{P}_i(t)\right]^\gamma,\ [P_i(t)]^\gamma = e^{[Q_i]^\gamma t}$$

with the Kolmogorov equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}[P_i(t)]^\gamma = [Q_i]^\gamma \times [P_i(t)]^\gamma.$$

Next, define the composite matrices as

$$Q_i := \left[[Q_i]^1, [Q_i]^2, \ldots, [Q_i]^{k_{\mathcal{N}_i}}\right] \in \mathcal{R}_{k_i\times k_i k_{\mathcal{N}_i}}$$

$$e^{Q_i t} := \left[e^{[Q_i]^1 t}, e^{[Q_i]^2 t}, \ldots, e^{[Q_i]^{k_{\mathcal{N}_i}} t}\right] \in \mathcal{R}_{k_i\times k_i k_{\mathcal{N}_i}}$$

$$P_i(t) := \left[[P_i(t)]^1, [P_i(t)]^2, \ldots, [P_i(t)]^{k_{\mathcal{N}_i}}\right] \in \mathcal{P}_{k_i\times k_i k_{\mathcal{N}_i}}.$$

Consequently, the PDV $\mathbf{p}_i(t) = \mathbb{E}\{\vec{x}_i(t)\}$ with $\vec{x}_i(t) := \delta_{k_i}^{\vec{x}_i(t)}$ evolves according to the ordinary differential equation

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}_i(t)\} = \left(Q_i \otimes I_{k_i k_{\mathcal{N}_i}}\right)\mathbb{E}\left\{\left(\vec{x}_{\mathcal{N}_i}(t)\otimes I_{k_i}\right)\vec{x}_i(t)\right\}\\ \mathbb{E}\{\vec{x}(0)\} = \mathbf{p}_0 \in \mathcal{P}^{\hat{k}}\end{cases} \tag{9}$$

with $\hat{k} := \prod_{i=1}^{n} k_i$ and $\vec{x}(t) := (\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t))$. In the following, we denote such systems as CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$.

D. Continuous-Time Control Markov Networks (CT-CMNs)

Consider a CT-CMN m control inputs over network structure $\mathcal{G}^c = (\mathcal{N}^c, \mathcal{E}^c)$. The system extends CT-MN properties to incorporate control input.

1) *Controlled Coupling Markov Property:* For each component $i \in \mathcal{N}$, the transition probability of $x_i \in \mathscr{D}_{k_i}$ depends on both neighbor state $x_{\mathcal{N}_i} \in \mathscr{D}_{k_{\mathcal{N}_i}}$ and control input $u \in \mathscr{D}_q$ through TPM $P_i^c(t) \in P_{k_i \times q k_i k_{\mathcal{N}_i}}$, defined as

$$[P_i^c(t)]_{\alpha,\beta}^{\gamma,\eta} = \Pr\left\{x_i(\theta+t) = \alpha | x_i(\theta) = \beta, x_{\mathcal{N}_i}(\theta) = \gamma, u(\theta) = \eta\right\}, \tag{10}$$

where $\alpha, \beta \in \mathscr{D}_{k_i}$, $\gamma \in \mathscr{D}_{k_{\mathcal{N}_i}}$, and $\eta \in \mathscr{D}_q$.

2) *Controlled Conditional Independence Property:* Component states $x_1(t+\theta), x_2(t+\theta), \ldots, x_n(t+\theta)$ for $t > 0, \theta \geq 0$, evolve conditionally independent on current state $x(\theta)$ and current input $u(\theta)$ as follows:

$$\Pr\{x(t+\theta)|x(\theta), u(\theta)\} = \prod_{i=1}^{n} \Pr\{x_i(t+\theta)|x(\theta), u(\theta)\}. \tag{11}$$

In particular, CT-CMNs can be viewed as switched CT-MNs with $q \cdot k_{\mathcal{N}_i}$ modes, where neighbor-control pair $(x_{\mathcal{N}_i}, u) \in \prod_{j\in\mathcal{N}_i} \mathscr{D}_{k_j} \times \mathscr{D}_q$ is the switching signal. Note that all properties of a CT-MC hold for each mode described by $[P_i^c(t)]^{\gamma,\eta} \in \mathcal{P}_{k_i \times k_i}$ with $(\gamma, \eta) \in \prod_{j\in\mathcal{N}_i} \mathscr{D}_{k_j} \times \mathscr{D}_q$. Then, the TRM $[Q_i^c]^{\gamma,\eta} \in \mathcal{R}_{k_i \times k_i}$ for each component $i \in \mathcal{N}$ regarding $(x_{\mathcal{N}_i}(\theta), u(\theta)) = (\gamma, \eta)$ is given by

$$[Q_i^c]^{\gamma,\eta} = \lim_{t\to 0^+} [\dot{P}_i^c(t)]^{\gamma,\eta}.$$

Then, the following Kolmogorov equation holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}[P_i^c(t)]^{\gamma,\eta} = [Q_i^c]^{\gamma,\eta} \times [P_i^c(t)]^{\gamma,\eta}$$

with the solution $[P_i^c(t)]^{\gamma,\eta} = e^{[Q_i^c]^{\gamma,\eta}t}$.

Given $\vec{x}_{\mathcal{N}_i}(t) := \delta_{k_{\mathcal{N}_i}}^{x_{\mathcal{N}_i}(t)} = \delta_{k_{\mathcal{N}_i}}^{\gamma}$ and $\vec{u}(t) := \delta_q^{u(t)} = \delta_q^{\eta}$, the dynamics of PDV $\mathbb{E}\{\vec{x}_i(t)\}$ for component $i \in \mathcal{N}$ is governed by

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left\{\vec{x}_i(t)\right\} = [Q_i^c]^{\gamma,\eta} \times \mathbb{E}\left\{\vec{x}_i(t)\right\}\\ \quad = Q_i^c\left(\vec{u}(t) \otimes I_{k_i k_{\mathcal{N}_i}}\right)\mathbb{E}\left\{\left(\vec{x}_{\mathcal{N}_i}(t) \otimes I_{k_i}\right)\vec{x}_i(t)\right\}\\ \mathbb{E}\{\vec{x}(0)\} = \mathbf{p}_0 \in \mathcal{P}^{\hat{k}}\end{cases} \tag{12}$$

with

$$Q_i^c := \left[[Q_i^c]^{1,1}, \ldots, [Q_i^c]^{k_{\mathcal{N}_i},1}, \ldots, [Q_i^c]^{k_{\mathcal{N}_i},q}\right] \in \mathcal{R}_{k_i \times q k_i k_{\mathcal{N}_i}}.$$

In the following, we concisely denote such systems as CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$.

E. Problem Formulation

Although CT-MC $\{Q\}$ and CT-CMC $\{Q^c\}$ have been extensively studied in [17], [18], [19], [21], and [27], the analysis of CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ and CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ remain largely unexplored. The core difficulty lies in characterizing the dynamics of the augmented state variable

$$\mathbb{E}\left\{\left(\vec{x}_1 \otimes I_{k_2\ldots k_n}\right)\left(\vec{x}_2 \otimes I_{k_3\ldots k_n}\right)\cdots\vec{x}_n\right\}.$$

From a technical perspective, this challenge arises from the following two key properties of expectation operators in this context:

$$\mathbb{E}\left\{\left(\vec{x}_1 \otimes I_{k_2\ldots k_n}\right)\left(\vec{x}_2 \otimes I_{k_3\ldots k_n}\right)\cdots\vec{x}_n\right\} \neq \left(\mathbb{E}\{\vec{x}_1\} \otimes I_{k_2\ldots k_n}\right)\left(\mathbb{E}\left\{\vec{x}_2\right\} \otimes I_{k_3\ldots k_n}\right)\cdots\mathbb{E}\{\vec{x}_n\} \tag{13}$$

and

$$\left(\mathbb{E}\left\{\vec{x}\right\} \otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}\right\} \neq \mathbf{M}_{\hat{k}}\mathbb{E}\left\{\vec{x}\right\} \tag{14}$$

where $\mathbf{M}_{\hat{k}} \in \mathcal{R}_{\hat{k}^2 \times \hat{k}}$ is the power-reducing matrix. It renders the conventional analysis frameworks (see, e.g., [19], [24], and[25]) invalid in the continuous-time setup.

Especially, in deterministic cases where probability vectors degrade into logical vectors, the dynamics of the augmented state variable $\vec{x} = (\vec{x}_1 \otimes I_{k_2 k_3\cdots k_n})(\vec{x}_2 \otimes I_{k_3\cdots k_n})\cdots\vec{x}_n$ can be simplified to

$$\vec{x}(t+1) = P_1\left(I_{k_1} \otimes P_2\right)\cdots\left(I_{k_1 k_2\cdots k_{n-1}} \otimes P_n\right)\vec{x}(t).$$

However, this equivalence fails for CT-MNs and CT-CMNs, leaving the existing methods (see, e.g., [19] and [25]) inapplicable.

Motivated by both practical applications (e.g., resource sharing problem [31], queueing problem [32], and biological engineering [17], [18], [19], [27]) and theoretical needs, this work aims to address the following three fundamental problems.

1) *Compact Representation:* Establish conditions under which CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ and CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ admit a state-space representation, which is a comprehensive and concise representations analogous to (4) and (6).
2) *Control Design:* Develop Lyapunov-based control synthesis for CT-MNs and CT-CMNs using the state-space representation, expanding upon recent advancements [19] to stabilize trajectories towards the desired state sets.
3) *Coordinated Behaviors:* Solve the leader-follower synchronization and output regulation problems for CT-MN $\mathcal{G}\{Q_1, \ldots, Q_n\}$ and CT-CMN $\mathcal{G}^c\{Q_1^c, \ldots, Q_n^c\}$, aspects that have not been addressed in previous studies due to the challenges in establishing the essential fusing conditions.

III. State-Space Representation of CT-MNs and CT-CMNs

In this section, we establish the state-space representation for CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ and CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$. A critical challenge arises from fundamental differences between discrete-time and continuous-time

Markov models: traditional methods for DT-MCs and DT-CMCs fail to generalize to CT-MNs and CT-CMNs.

A. Compact Representation of CT-MNs

We first establish a foundational lemma for differentiating Khatri–Rao products of time-dependent matrices, which is essential for deriving the compact form of CT-MN $\mathcal{G}\{Q_1,\dots,Q_n\}$.

Lemma 1: Let $A(t)\in\mathcal{R}_{p\times n}$ and $B(t)\in\mathcal{R}_{q\times n}$ be differentiable matrix-valued functions of $t\geq 0$. Then, the time derivative of their Khatri–Rao product satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[A(t)*B(t)\right]=\frac{\mathrm{d}A(t)}{\mathrm{d}t}*B(t)+A(t)*\frac{\mathrm{d}B(t)}{\mathrm{d}t}. \tag{15}$$

Proof: Define $C(t):=A(t)*B(t)\in\mathcal{R}_{pq\times n}$. Let $A_i(t)$, $B_i(t)$, and $C_i(t)$ represent the ith columns of matrices $A(t):=(a_{i,j}(t))_{p\times n}$, $B(t):=(b_{i,j}(t))_{q\times n}$, and $C(t):=(c_{i,j}(t))_{pq\times n}$, respectively. By the definition of the Khatri–Rao product, one has

$$C_i(t)=A_i(t)\otimes B_i(t)=\begin{bmatrix}a_{1,i}(t)B_i(t)\\a_{2,i}(t)B_i(t)\\\vdots\\a_{p,i}(t)B_i(t)\end{bmatrix},\forall i\in\mathscr{D}_n.$$

Differentiating $C(t)$ yield

$$\dot{C}_i(t)=\frac{\mathrm{d}}{\mathrm{d}t}[A_i(t)\otimes B_i(t)]=\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}a_{i,1}(t)B_i(t)\\a_{i,2}(t)B_i(t)\\\vdots\\a_{i,p}(t)B_i(t)\end{bmatrix}$$
$$=\begin{bmatrix}\dot{a}_{i,1}(t)B_i(t)+a_{i,1}(t)\dot{B}_i(t)\\\dot{a}_{i,2}(t)B_i(t)+a_{i,2}(t)\dot{B}_i(t)\\\vdots\\\dot{a}_{i,p}(t)B_i(t)+a_{i,p}(t)\dot{B}_i(t)\end{bmatrix}$$
$$=\dot{A}_i(t)\otimes B_i(t)+A_i(t)\otimes\dot{B}_i(t).$$

Finally, aggregating over all columns $i\in\mathscr{D}_n$ gives $\dot{C}(t)=\dot{A}(t)*B(t)+A(t)*\dot{B}(t)$. The proof is complete. ■

Without loss of generality, we assume that the neighbors of component $i\in\mathcal{N}$ are represented as

$$\mathcal{N}_i=\{r_1,r_2,\dots,r_{n_i}\,|\,r_1<r_2<\cdots<r_{n_i}\}.$$

To rearrange the order of components $i\in\mathcal{N}$ and its neighbors $\mathcal{N}_i$ in the natural order, we define the matrix $\mathbf{W}_{\mathcal{N}_i}\in\mathcal{R}_{k_ik_{\mathcal{N}_i}\times k_ik_{\mathcal{N}_i}}$ for x_i as

$$\mathbf{W}_{\mathcal{N}_i}=\begin{cases}I_{k_ik_{\mathcal{N}_i}}, & r_{n_i}<i\\ \mathbf{W}_{[r_1,\dots,i,\dots,r_{n_i}]}, & r_1<i<r_{n_i}\\ \mathbf{W}_{[i,r_1,\dots,r_{n_i}]}, & r_1>i\end{cases}$$

with $\mathbf{W}_{[r_1,\dots,r_{n_i}]}$ being the permutation matrix as defined in [25]. Let

$$\Phi_i:=\bigotimes_{j=1}^{n}\phi_j=\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_n\in\mathcal{R}_{k_ik_{\mathcal{N}_i}\times\hat{k}}$$

with

$$\phi_j=\begin{cases}I_{k_j}, & j\in\mathcal{N}_i\cup\{i\}\\ \mathbf{1}_{k_j}^{\top}, & j\notin\mathcal{N}_i\cup\{i\}.\end{cases}$$

The following theorem indicates that, for CT-MNs satisfying the coupling Markov property and conditional independence property, their compact TRM can be computed accordingly.

Theorem 1: For any CT-MN $\mathcal{G}\{Q_1,Q_2,\dots,Q_n\}$ satisfying the coupling Markov property and conditional independence property, there exists an equivalent CT-MC characterized by the TRM

$$\mathbf{Q}:=\sum_{i=1}^{n}S_{[i,1]}*S_{[i,2]}*\cdots*S_{[i,n]}\in\mathcal{R}_{\hat{k}\times\hat{k}} \tag{16}$$

where

$$S_{[i,j]}:=\begin{cases}Q_i\Phi_i\mathbf{W}_{\mathcal{N}_i}, & j=i\\ (\mathbf{1}_{k_{\mathcal{N}_j}}^{\top}\otimes I_{k_j})\Phi_j\mathbf{W}_{\mathcal{N}_j}, & j\neq i.\end{cases} \tag{17}$$

Proof: To prove this theorem, we first demonstrate that a given CT-MN $\mathcal{G}\{Q_1,Q_2,\dots,Q_n\}$ is equivalent to a CT-MC characterized by TPM $\mathbf{P}(t)$. Based on this, the TRM $\mathbf{Q}$ can be calculated using $\mathbf{Q}=\lim_{t\to 0^+}\dot{\mathbf{P}}(t)$.

For each component $i\in\mathcal{N}$, the PDV $\mathbb{E}\{\vec{x}_i(t+\theta)\}$ with $t,\theta\geq 0$ evolves as

$$\mathbb{E}\left\{\vec{x}_i(t+\theta)\right\}=P_i(t)\mathbb{E}\left\{\left(\vec{x}_{\mathcal{N}_i}(\theta)\otimes I_{k_i}\right)\vec{x}_i(\theta)\right\}. \tag{18}$$

Applying neighborhood permutation matrix $\mathbf{W}_{\mathcal{N}_i}$ and coupling operator Φ_i, one can obtain

$$\begin{aligned}\mathbb{E}\left\{\vec{x}_i(t+\theta)\right\}&=P_i(t)\mathbf{W}_{\mathcal{N}_i}\mathbb{E}\left\{\vec{x}_{\mathcal{N}_i\cup\{i\}}(\theta)\right\}\\&=P_i(t)\Phi_i\mathbf{W}_{\mathcal{N}_i}\mathbb{E}\left\{\vec{x}(\theta)\right\}\\&:=\hat{P}_i(t)\mathbb{E}\left\{\vec{x}(\theta)\right\}\end{aligned} \tag{19}$$

with $\hat{P}_i(t):=P_i(t)\Phi_i\mathbf{W}_{\mathcal{N}_i}$. Then, by the conditional independence property, the probability distribution of state variable $x(t+\theta)=(a_1,a_2,\dots,a_n)^{\top}$ satisfies

$$\begin{aligned}&\Pr\left\{x(t+\theta)=(a_1,\dots,a_n)^{\top}\right\}\\&=\sum_{p\in\prod_{i=1}^{n}\mathscr{D}_{k_i}}\Pr\left\{x(t+\theta)=(a_1,\dots,a_n)^{\top}|x(\theta)=p\right\}\Pr\left\{x(\theta)=p\right\}\\&=\sum_{p\in\prod_{i=1}^{n}\mathscr{D}_{k_i}}\Pr\left\{x(\theta)=p\right\}\left(\prod_{i=1}^{n}\Pr\left\{x_i(t+\theta)=a_i|x(\theta)=p\right\}\right)\\&=\mathrm{Row}_{(a_1,\dots,a_n)}\left(\hat{P}_1(t)*\hat{P}_2(t)*\cdots*\hat{P}_n(t)\right)\mathbb{E}\left\{\vec{x}(\theta)\right\}.\end{aligned} \tag{20}$$

The last equality holds owing to

$$\left[\hat{P}_i(t)\right]_{a_i,p}=\Pr\{x_i(t+\theta)=a_i|x(\theta)=p\},\ i\in\mathcal{N}$$

and

$$\begin{aligned}&\mathrm{Row}_{(a_1,\dots,a_n)}\left(\hat{P}_1(t)*\hat{P}_2(t)*\cdots*\hat{P}_n(t)\right)\\&=\mathrm{Row}_{a_1}\left[\hat{P}_1(t)\right]\otimes\mathrm{Row}_{a_2}\left[\hat{P}_2(t)\right]\otimes\cdots\otimes\mathrm{Row}_{a_n}\left[\hat{P}_n(t)\right].\end{aligned}$$

Formula (20) connects the probability distribution of random variable $x(t+\theta) = (a_1, a_2, \ldots, a_n)^\top$ under TRMs $\hat{P}_i(t), i \in \mathcal{N}$. We can convert the considered CT-MN into an equivalent CT-MC as follows:

$$\mathbb{E}\left\{\vec{x}(t+\theta)\right\} = \mathbf{P}(t)\mathbb{E}\left\{\vec{x}(\theta)\right\} \tag{21}$$

with

$$\mathbf{P}(t) = \hat{P}_1(t) * \hat{P}_2(t) * \cdots * \hat{P}_n(t).$$

The aggregated TPM $\mathbf{P}(t)$ yields an equivalent CT-MC over the state space $\mathscr{D}_{\hat{k}}$.

Differentiating $\mathbf{P}(t)$ and applying Lemma 1 gives the TRM of the equivalent CT-MC as follows:

$$\begin{aligned}
\mathbf{Q} &= \lim_{t\to 0^+} \dot{\mathbf{P}}(t) \\
&= \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{P}_1(t) * \hat{P}_2(t) * \cdots * \hat{P}_n(t)\right] \\
&= \lim_{t\to 0^+} \sum_{i=1}^{n} \hat{P}_1(t) * \cdots * \left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_i(t)\right] * \cdots * \hat{P}_n(t) \\
&= \sum_{i=1}^{n} \lim_{t\to 0^+} \hat{P}_1(t) * \cdots * \left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_i(t)\right] * \cdots * \hat{P}_n(t) \\
&= \sum_{i=1}^{n} S_{[i,1]} * S_{[i,2]} * \cdots * S_{[i,n]}
\end{aligned}$$

where $S_{[i,j]}$ captures both local dynamics and neighborhood interactions. Therefore, the proof is complete. ■

Theorem 1 establishes that under the coupling Markov property and conditional independence property, CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ admits an exact representation as a CT-MC $\{\mathbf{Q}\}$ with dynamics governed by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}(t)\} = \left(\mathbf{Q} \otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}(t)\right\} \\ \mathbb{E}\{\vec{x}(0)\} = \mathbf{p}_0 \in \mathcal{P}^{\hat{k}}. \end{cases}$$

The state-space representation in (16) provides a foundation for adapting the existing CT-MC analysis methods to study CT-MN behaviors.

B. Compact Representation of CT-CMNs

Proceeding forward, we delve into the amalgamation of CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ as outlined in (12). It is worth noting that if the dynamics of this CT-CMN do not adhere to (12), it can be transformed accordingly using the permutation matrix and dummy matrix.

Theorem 2: For any CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ satisfying the controlled coupling Markov property and conditional independence property, there exists an equivalent CT-CMC characterized by the TRM as follows:

$$\mathbf{Q}^c := \sum_{i=1}^{n} S^c_{[i,1]} * S^c_{[i,2]} * \cdots * S^c_{[i,n]} \in \mathcal{R}_{\hat{k}\times q\hat{k}} \tag{22}$$

where

$$S^c_{[i,j]} := \begin{cases} Q_i^c\left[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})\right], & j = i \\ \left[\mathbf{1}^\top_{qk_{\mathcal{N}_j}} \otimes I_{k_j}\right]\left[I_q \otimes (\Phi_j \mathbf{W}_{\mathcal{N}_j})\right], & j \neq i. \end{cases} \tag{23}$$

Proof: Similar to Theorem 1, we begin by establishing that CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ is equivalent to a CT-CMC characterized by TPM $\mathbf{P}^c(t)$. Subsequently, the TRM of the equivalent CT-MC can be calculated as $\mathbf{Q}^c = \lim_{t\to 0^+} \dot{\mathbf{P}}^c(t)$.

Considering CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ in (12), where the TPM of component $i \in \mathcal{N}$ is denoted by $P_i^c(t)$, the PDV $\mathbb{E}\{\vec{x}_i(t+\theta)\}$ satisfies

$$\begin{aligned}
\mathbb{E}\left\{\vec{x}_i(t+\theta)\right\} = P_i^c(t)&\left(\vec{u}(\theta) \otimes I_{k_i k_{\mathcal{N}_i}}\right) \\
&\cdot \mathbb{E}\left\{\left(\vec{x}_{\mathcal{N}_i}(\theta) \otimes I_{k_i}\right)\vec{x}_i(\theta)\right\}
\end{aligned} \tag{24}$$

where control input $u(t)$ can be considered as a switching signal. Following a similar derivation as in (19), formula (24) can be equivalently represented as

$$\begin{aligned}
&\mathbb{E}\{\vec{x}_i(t+\theta)\} \\
&= P_i^c(t)\left(\vec{u}(\theta) \otimes I_{\hat{k}}\right)\Phi_i \mathbf{W}_{\mathcal{N}_i}\mathbb{E}\left\{\vec{x}(\theta)\right\} \\
&= P_i^c(t)\left[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})\right]\left(\vec{u}(\theta) \otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}(\theta)\right\} \\
&:= \hat{P}_i^c(t)\left(\vec{u}(\theta) \otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}(\theta)\right\}.
\end{aligned} \tag{25}$$

Here, $\hat{P}_i^c(t) := P_i^c(t)[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})] \in \mathcal{R}_{k_i\times q\hat{k}}$. Consequently, by the controlled conditional independence property, the probability distribution of $x(t+\theta) = (a_1, a_2, \ldots, a_n)^\top$ satisfies

$$\begin{aligned}
&\Pr\left\{x(t+\theta) = (a_1, \ldots, a_n)^\top\right\} \\
&= \sum_{p,r}\Bigg[\Pr\left\{x(t+\theta) = (a_1, \ldots, a_n)^\top | x(\theta) = p, u(\theta) = r\right\} \\
&\qquad \cdot \Pr\left\{x(\theta) = p, u(\theta) = r\right\}\Bigg] \\
&= \sum_{p,r}\Bigg[\left(\prod_{j=1}^{n}\Pr\left\{x_j(t+\theta) = a_j | x(\theta) = p, u(\theta) = r\right\}\right) \\
&\qquad \cdot \Pr\left\{x(\theta) = p, u(\theta) = r\right\}\Bigg] \\
&= \mathrm{Row}_{(a_1,\ldots,a_n)}\left(\hat{P}_1^c(t) * \cdots * \hat{P}_n^c(t)\right)\left(\vec{u}(\theta) \otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}(\theta)\right\}.
\end{aligned}$$

Letting $\mathbf{P}^c(t) := \hat{P}_1^c(t) * \hat{P}_2^c(t) * \cdots * \hat{P}_n^c(t)$, one has

$$\mathbb{E}\left\{\vec{x}(t+\theta)\right\} = \mathbf{P}^c(t)\left(\vec{u}(t) \otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}(\theta)\right\}. \tag{26}$$

The aggregated TPM $\mathbf{P}^c$ yields an equivalent CT-CMC over the state space $\prod_{i\in\mathcal{N}} \mathscr{D}_{k_i}$.

Then, differentiating $\mathbf{P}^c(t)$ and applying Lemma 1 yields the TRM of the equivalent CT-CMC as follows:

$$\begin{aligned}
\mathbf{Q}^c &= \lim_{t\to 0^+} \dot{\mathbf{P}}^c(t) \\
&= \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{P}_1^c(t) * \hat{P}_2^c(t) * \cdots * \hat{P}_n^c(t)\right] \\
&= \lim_{t\to 0^+} \sum_{i=1}^{n} \hat{P}_1^c(t) * \cdots * \left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_i^c(t)\right] * \cdots * \hat{P}_n^c(t)
\end{aligned}$$

$$= \sum_{i=1}^{n} \lim_{t\to 0^+} \hat{P}_1^c(t) * \cdots * \left[\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_i^c(t)\right] * \cdots * \hat{P}_n^c(t). \quad (27)$$

Moreover, from (25), we have

$$\lim_{t\to 0^+} \hat{P}_i^c(t) = \left[\mathbf{1}_{qk_{\mathcal{N}_i}}^\top \otimes I_{k_i}\right][I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})]$$
$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_i^c(t) = Q_i^c\left[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})\right]. \quad (28)$$

Incorporating (28) into the final formula of (27), one can derive (22). This completes the proof. ■

The intuitive nature of Theorem 2 becomes evident through (23). When $i = j$, the resulting expression follows a compelling pattern:

$$\begin{aligned} S_{[i,j]}^c &= \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_i^c(t) \\ &= \lim_{t\to 0^+} \frac{\mathrm{d}P_i^c(t)}{\mathrm{d}t}\cdot[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})] \\ &= Q_i^c\left[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})\right]. \end{aligned}$$

Here, $\mathbf{W}_{\mathcal{N}_i}$ orchestrates the natural order of component i and its neighbors, while Φ_i serves as a dummy matrix, defining

$$\mathbb{E}\left\{\vec{x}_{\mathcal{N}_i \bigcup\{i\}}(\theta)\right\} = \Phi_i \mathbb{E}\left\{\vec{x}(\theta)\right\}.$$

Conversely, if $i \neq j$, it has

$$\begin{aligned} S_{[i,j]}^c &= \lim_{t\to 0^+} \hat{P}_i^c(t) \\ &= \lim_{t\to 0^+} P_i^c(t)\left[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})\right] \\ &= \left(\mathbf{1}_{qk_{\mathcal{N}_j}}^\top \otimes I_{k_j}\right)\left[I_q \otimes (\Phi_i \mathbf{W}_{\mathcal{N}_i})\right]. \end{aligned}$$

In essence, Theorem 2 asserts that, under the coupling Markov property and conditional independence property, CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ is equivalent to CT-CMC $\{\mathbf{Q}^c\}$. The resultant PDV adheres to the following equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}(t)\} = \mathbf{Q}^c\left(\vec{u}(t)\otimes I_{\hat{k}}\right)\mathbb{E}\left\{\vec{x}(t)\right\} \\ \mathbb{E}\{\vec{x}(0)\} = \mathbf{p}_0 \in \mathcal{P}^{\hat{k}}. \end{cases}$$

Given the parallels with the format found in [25], we refer to (22) as the state-space representation of CT-CMNs hereafter. Based on this, all the existing approaches to CT-CMCs can be adapted or extended to explore the dynamic behaviors of CT-CMN $\mathcal{G}\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$.

C. Relation Between CT-MNs and DT-MNs

In this subsection, we explore the relationships between CT-MNs and DT-MNs, following the methodology outlined in [26] and building upon the recent insights from [19]. Initiating with CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$, we discretize its state trajectory by fixed-interval sampling at τ time instants. At each sampling point, the state of each component $i \in \mathcal{N}$ is captured by the random variable $x_i(t\tau), i \in \mathcal{N}$. The mathematical expectation at the tth sampling instant is denoted as

$$\mathbf{z}_i^\tau(t) := \mathbf{p}_i(t\tau) = \mathbb{E}\{x_i(t\tau)\},\ t \in \mathbb{N}_+.$$

It follows a dynamic evolution described by

$$\begin{cases} \vec{\mathbf{z}}_1^\tau(t+1) = e^{Q_1\tau}\vec{\mathbf{z}}^\tau(t) \\ \vec{\mathbf{z}}_2^\tau(t+1) = e^{Q_2\tau}\vec{\mathbf{z}}^\tau(t) \\ \vdots \\ \vec{\mathbf{z}}_n^\tau(t+1) = e^{Q_n\tau}\vec{\mathbf{z}}^\tau(t) \end{cases} \quad (29)$$

with $\vec{\mathbf{z}}_i^\tau(t) := \delta_{k_i}^{\mathbf{z}_i^\tau(t)}$, $i \in \mathcal{N}$, and $\vec{\mathbf{z}}^\tau(t) := \mathbb{E}\{\vec{x}(t\tau)\}$. This discrete-time evolution can be viewed as a DT-MN.

According to [26], there are two ways to obtain the state-space representation of DT-MN (29). The first approach integrates the conditional independence condition: random variables $x_1((t+1)\tau), x_2((t+1)\tau), \ldots, x_n((t+1)\tau)$ are conditionally independent on $x(t\tau)$ for all $t \geq 0$. This leads to the following conclusion.

Lemma 2 (See [26]): Consider DT-MN (29). If the conditional independence condition is satisfied, then the state-space representation of (29) is in the form of

$$\vec{\mathbf{z}}^\tau(t+1) = \mathrm{P}_\tau \vec{\mathbf{z}}^\tau(t)$$

where $\mathrm{P}_\tau = e^{Q_1\tau} * e^{Q_2\tau} * \cdots * e^{Q_n\tau}$.

The second relies on the independence condition and is articulated as follows.

Lemma 3: Consider DT-MN (29). If the independence condition is satisfied, i.e., $x_1(t\tau), x_2(t\tau), \ldots, x_n(t\tau)$ are mutually independent, and the following criterion holds:

$$H\mathbf{R}_{\hat{k}}^{n-1}\xi = H\xi^n, \forall \xi \in \mathcal{P}^n \quad (30)$$

then

$$\vec{\mathbf{z}}^\tau(t+1) = \mathrm{P}_\tau \vec{\mathbf{z}}^\tau(t)$$

where $\mathbf{R}_{\hat{k}} := [\delta_{\hat{k}}^1 \otimes \delta_{\hat{k}}^1, \delta_{\hat{k}}^2 \otimes \delta_{\hat{k}}^2, \ldots, \delta_{\hat{k}}^{\hat{k}} \otimes \delta_{\hat{k}}^{\hat{k}}]$ and $H := \prod_{j=1}^n (I_{\hat{k}^{j-1}} \otimes e^{Q_j\tau} \otimes I_{\hat{k}^{n-j}})$.

Lemmas 2 and 3 elucidate that the state-space representation of CT-MNs and DT-MNs can be derived under either the conditional independence condition or criterion (30). When only the independence condition is satisfied, DT-MN (29) follows that

$$\begin{aligned} \vec{\mathbf{z}}^\tau(t+1) &= \prod_{j=1}^n \left(I_{\hat{k}^{j-1}} \otimes \vec{\mathbf{z}}_j^\tau(t+1) \otimes I_{\hat{k}^{n-j}}\right) \\ &= \prod_{j=1}^n \left(I_{\hat{k}^{j-1}} \otimes \left(e^{Q_j\tau}\vec{\mathbf{z}}_j^\tau(t)\right) \otimes I_{\hat{k}^{n-j}}\right) \\ &= H[\vec{\mathbf{z}}^\tau(t)]^n \end{aligned}$$

resulting in a heterogeneous Markov chain. For further insights on the relationship between the independent and conditionally independent models, refer to [26].

Conclusively, the following Theorem 3 reveals that the TRM $\mathbf{Q}$ of a CT-MN can be interpreted as the differential limit of TPM P_τ in its sampling network as the sampling period τ tends to zero.

Theorem 3: The CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ and its corresponding DT-MN $\{\mathbf{Q}\}$ satisfies

$$\mathbf{Q} = \lim_{\tau\to 0^+} \dot{P}_\tau \quad (31)$$

where $\mathbf{Q}$ is defined in (16).

D. Relations With Existing Models

The state-space representation of CT-MNs and CT-CMNs in this article is a unified framework for CT-MCs and CT-CMCs. Next, we discuss the inclusion relations between CT-MNs and some existing models.

Our model shares a mathematical description similar to the CT-MNs in [29] and [30], commonly employed in probabilistic reasoning and learning theory. However, a notable distinction lies in their update mechanisms. Existing models typically assume asynchronous updates, where only one variable transitions at any time instant. In contrast, our framework caters to both synchronous and asynchronous processes, rendering it more available. Consequently, our established conditions are applicable to the networks in [29] and [30].

Moreover, prior work on continuous-time Boolean networks [33] and continuous-time logical networks [17] often ignores network topology and logical constraints, complicating the analysis of multicomponent swarm behaviors. Notably, Theorems 1 and 2 furnish explicit state-space representations for CT-MNs and CT-CMNs, facilitating their direct application in such systems.

In particular, our model also serves as a generalization of stochastic automata networks [31], [32], [34], widely utilized in parallel and distributed computer systems. A stochastic automata network comprises interacting but largely independent automata [32], constituting a special case of our CT-MNs where components evolve autonomously. Importantly, while stochastic automata networks focus exclusively on independent models (a subclass of our CT-MNs), our tensor-based representation enables the direct computation of their state-space dynamics [34]. Specifically, we express the TRM of a CT-MN as tensor operations on automata-specific matrices. In this context, the TRM $\mathbf{Q}$ of an independent stochastic automata network is formulated as

$$\mathbf{Q} = \bigoplus_{i=1}^{n} Q_i \tag{32}$$

where operation $\oplus$ denote the tensor sum defined in [34], matrix $\mathbf{Q}$ is referred to as the global infinitesimal generator, and Q_i signifies the local infinitesimal generator according to [31], [32], [34]. Indeed, (32) can be deduced from (16) in Theorem 1.

Theorem 4: For independent stochastic automata networks, (32) is equivalent to (16).

Proof: Consider a network $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ of independent stochastic automatas with the dynamics of PDV obeying

$$\dot{\mathbf{p}}_i(t) = Q_i \mathbf{p}_i(t), \forall i \in \mathcal{N}$$

where $\mathbf{p}_i(t) = \mathbb{E}\{\vec{x}_i(t)\}$. It yields

$$\mathbf{p}_i(t+\theta) = P_i(t)\mathbf{p}_i(\theta)$$

with $P_i(t) = e^{Q_i t}$. As stochastic automata networks are independent, we derive

$$\begin{aligned} \mathbf{p}(t+\theta) &= \prod_{i=1}^{n} \left(I_{\tilde{k}_{i-1}} \otimes \mathbf{p}_i(t+\theta) \otimes I_{\tilde{k}_{n-i}} \right) \\ &= \prod_{i=1}^{n} \left(I_{\tilde{k}_{i-1}} \otimes (P_i(t)\mathbf{p}_i(\theta)) \otimes I_{\tilde{k}_{n-i}} \right) \\ &= \left(\bigotimes_{i=1}^{n} P_i(t) \right) \mathbf{p}(\theta) \end{aligned} \tag{33}$$

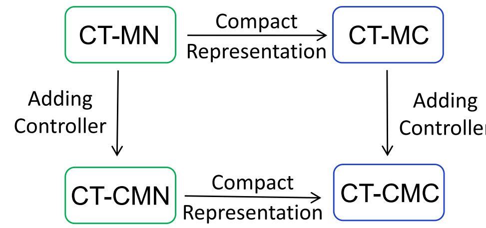

Fig. 1. Relations between CT-MNs, CT-CMNs, CT-MCs, and CT-CMCs.

where $\tilde{k}_\alpha = k_1 k_2 \cdots k_\alpha$. Consequently, one can obtain

$$P(t) = \bigotimes_{i=1}^{n} P_i(t)$$

and

$$\mathbf{Q} = \lim_{t \to 0^+} \dot{P}(t) = \lim_{t \to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} \left[\otimes_{i=1}^{n} P_i(t) \right] = \bigoplus_{i=1}^{n} Q_i.$$

Therefore, the proof is complete. ■

Fig. 1 summarizes the relationships among CT-MNs, CT-CMNs, CT-MCs, and CT-CMCs.

IV. Stabilization via Control Lyapunov Functions

This section investigates two fundamental problems in dynamic systems: global stability and global stabilization. Building on the state-space representation of CT-MNs and CT-CMNs established in Section III, we now examine their behavioral control through CT-CMCs characterized by TRM $\mathbf{Q}^c \in \mathcal{R}_{\hat{k} \times q\hat{k}}$.

By Theorem 1, CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ is equivalent to CT-MC $\{\mathbf{Q}\}$ with TRM $\mathbf{Q}$ calculated by (16). It yields the system dynamics of CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ as follows:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}(t)\} = \mathbf{Q}\mathbb{E}\{\vec{x}(t)\} \\ \mathbb{E}\{\vec{x}(0)\} = \mathbf{p}_0. \end{cases}$$

The stability analysis of CT-MCs has been thoroughly characterized through Lyapunov functions in [18] and [19]. Consequently, the stability of CT-MN $\mathcal{G}\{Q_1, Q_2, \ldots, Q_n\}$ and CT-CMN $\mathcal{G}^c\{Q_1^c, Q_2^c, \ldots, Q_n^c\}$ can be determined via the Lyapunov functions constructed for their equivalent CT-MC $\{\mathbf{Q}\}$ and CT-CMC $\{\mathbf{Q}^c\}$, respectively.

Therefore, we focus on the set stability of CT-MC $\{\mathbf{Q}\}$ and CT-CMC $\{\mathbf{Q}^c\}$, with particular applications to leader-follower synchronization and output regulation as detailed in Section V. While recent work [19] employed Lyapunov functions primarily for convergence analysis, we extend this framework to controller design. For completeness, we first review the foundational Lyapunov theory from [19] before presenting our control-theoretic extensions.

A. Lyapunov Functions of CT-MCs

Building on the stability characterization of CT-MCs in [19], we formally review the key definitions and results that will serve as the foundation for the control design.

Definition 1 (See [19]): Consider CT-MC $\{\mathbf{Q}\}$ with target state $x_e \in \mathscr{D}_k$ and target state set $S_e \subseteq \mathscr{D}_k$.

1) It is said to be asymptotically x_e-stable in distribution if

$$\lim_{t\to+\infty} \Pr\{x(t;\mathbf{p}_0)=x_e\}=1, \forall \mathbf{p}_0\in\mathcal{P}^k.$$

2) It is said to be asymptotically S_e-stable in distribution if

$$\lim_{t\to+\infty} \Pr\{x(t;\mathbf{p}_0)\in S_e\}=1, \forall \mathbf{p}_0\in\mathcal{P}^k.$$

Definition 2 (See [19]): A function $V:\mathscr{D}_k\to\mathcal{R}^1$ is called a Lyapunov function of CT-MC $\{\mathbf{Q}\}$ subject to the given state $x_e\in\mathscr{D}_k$ if it satisfies the following conditions:

1) $V(x(t))=0$ for $x(t)=x_e$ and $V(x(t))>0$ for all $x(t)\in\mathscr{D}_k\backslash\{x_e\}$;
2) $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{V(x(t))\}=0$ for $x(t)=x_e$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{V(x(t))\}<0$ for all $x(t)\in\mathscr{D}_k\backslash\{x_e\}$.

Then, let $\vec{x}:=\delta_k^x$ and $\vec{x}_e:=\delta_k^{x_e}$ be the canonical basis vector representations of x and x_e, respectively. One can derive the following result.

Lemma 4 (See [19]): Given state $x_e\in\mathscr{D}_k$, CT-MC $\{\mathbf{Q}\}$ is asymptotically x_e-stable in distribution if and only if there exists a vector $\lambda\in\mathcal{R}^k$ satisfying

$$\lambda^\top\vec{x}_e=0$$
$$\lambda^\top\vec{x}>0,\ \forall x\in\mathscr{D}_k\backslash\{x_e\}$$
$$\lambda^\top\mathbf{Q}\vec{x}_e=0$$
$$\lambda^\top\mathbf{Q}\vec{x}<0,\ \forall x\in\mathscr{D}_k\backslash\{x_e\}.$$

Definition 3 (See [19]): A function $V:\mathscr{D}_k\to\mathcal{R}^1$ is called a Lyapunov function of CT-MC $\{\mathbf{Q}\}$ subject to the given state set $S_e\subseteq\mathscr{D}_k$ if it satisfies the following conditions:

1) $V(x(t))\geq 0$ for all $x(t)\in\mathscr{D}_k$;
2) $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{V(x(t))\}\leq 0$ for all $x(t)\in S_e$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{V(x(t))\}<0$ for all $x(t)\in\mathscr{D}_k\backslash S_e$.

Lemma 5 (See [19]): Given state set $S\subseteq\mathscr{D}_k$, CT-MC $\{\mathbf{Q}\}$ is asymptotically S-stable in distribution if and only if there exists a vector $\lambda\in\mathcal{R}^k$ satisfying

$$\lambda^\top\vec{x}\geq 0,\forall x\in\mathscr{D}_k$$
$$\lambda^\top\mathbf{Q}\vec{x}\leq 0,\forall x\in S$$
$$\lambda^\top\mathbf{Q}\vec{x}<0,\forall x\in\mathscr{D}_k\backslash S.$$

Based on Definitions 2 and 3, we construct the Lyapunov function for CT-MC $\{\mathbf{Q}\}$ as follows:

$$V(\vec{x})=\lambda^\top\vec{x} \tag{34}$$

where λ can be obtained from Lemmas 4 and 5, respectively. Moreover, the affine transformation

$$\tilde{V}(\vec{x})=(\lambda+a\mathbf{1}_k)^\top\vec{x}$$

also preserves the Lyapunov properties for any $a\in\mathcal{R}$.

B. Control Lyapunov Functions of CT-CMCs

The preceding results on Lyapunov functions for CT-MCs have been limited to uncontrolled systems. We now extend these results to develop control Lyapunov functions for CT-CMCs, beginning with key stabilization definitions.

Definition 4: Consider CT-CMC $\{\mathbf{Q}^c\}$ with target state $x_e\in\mathscr{D}_k$ and target state set $S_e\subseteq\mathscr{D}_k$.

1) It is said to be asymptotically x_e-stabilizable in distribution if there exists a state feedback controller $u(t)=\mathcal{K}(x(t))$ with $\mathcal{K}:\mathscr{D}_k\to\mathscr{D}_q$ such that

$$\lim_{t\to+\infty} \Pr\{x(t;u,\mathbf{p}_0)=x_e\}=1, \forall \mathbf{p}_0\in\mathcal{P}^k.$$

2) It is said to be asymptotically S_e-stabilizable in distribution if there exists a state feedback controller $u(t)=\mathcal{K}(x(t))$ with $\mathcal{K}:\mathscr{D}_k\to\mathscr{D}_q$ such that

$$\lim_{t\to+\infty} \Pr\{x(t;u,\mathbf{p}_0)\in S_e\}=1, \forall \mathbf{p}_0\in\mathcal{P}^k.$$

Using the canonical correspondence, the state feedback controller $u(t)=\mathcal{K}(x(t))$ admits the vector form

$$\vec{u}(t)=K\vec{x}(t) \tag{35}$$

where $K\in\mathcal{L}_{q\times k}$ is one-to-one correspondence with function $\mathcal{K}$. Then, the stabilizability criterion is established.

Proposition 1: Given the target state $x_e\in\mathscr{D}_k$. CT-CMC $\{\mathbf{Q}^c\}$ is asymptotically x_e-stablizable in distribution if and only if there exists a vector $\hat{\lambda}\in\mathcal{R}^k$ and a logical matrix $K\in\mathcal{L}_{q\times k}$ satisfying

$$\hat{\lambda}^\top\vec{x}_e=0 \tag{36a}$$
$$\hat{\lambda}^\top\vec{x}>0,\forall x\in\mathscr{D}_k\backslash\{x_e\} \tag{36b}$$
$$\hat{\lambda}^\top\mathbf{Q}^c\left(K\otimes I_k\right)\mathbf{M}_k\vec{x}_e=0 \tag{36c}$$
$$\hat{\lambda}^\top\mathbf{Q}^c\left(K\otimes I_k\right)\mathbf{M}_k\vec{x}<0,\forall x\in\mathscr{D}_k\backslash\{x_e\} \tag{36d}$$

where $\mathbf{M}_k\in\mathcal{R}_{k^2\times k}$ is the power-reducing matrix.

Proof: The dynamics of CT-CMC $\{\mathbf{Q}^c\}$ satisfies

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}(t)\}=\mathbf{Q}^c\left(\vec{u}(t)\otimes I_k\right)\mathbb{E}\left\{\vec{x}(t)\right\}\\ \mathbb{E}\{\vec{x}(0)\}=\mathbf{p}_0.\end{cases}$$

Under state feedback controller $\vec{u}(t)=K\vec{x}(t)$, this dynamics equation becomes the following closed-loop form:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}(t)\}=\mathbf{Q}^c\left(K\otimes I_k\right)\mathbf{M}_k\mathbb{E}\left\{\vec{x}(t)\right\}\\ \mathbb{E}\{\vec{x}(0)\}=\mathbf{p}_0\end{cases}$$

which describes the dynamics of CT-MC $\{\mathbf{Q}^c(K\otimes I_k)\mathbf{M}_k\}$. According to Lemma 4, CT-MC $\{\mathbf{Q}^c(K\otimes I_k)\mathbf{M}_k\}$ is asymptotically x_e-stable in distribution, implying that CT-CMC $\{\mathbf{Q}^c\}$ is asymptotically x_e-stablizable in distribution. ■

While Proposition 1 provides a straightforward way to verify stabilizability with given $\hat{\lambda}$ and matrix K, we now present a constructive procedure for designing state feedback stabilizers. We begin with single-state stabilization of CT-CMC $\{\mathbf{Q}^c\}$ by defining the admissible control set $\mathcal{H}_x\subseteq\Delta_q$ for each state $x\in\mathscr{D}_k$ as follows:

$$\mathcal{H}_x=\begin{cases}\left\{\delta_q^j\in\Delta_q\,|\,\mathbf{Q}^c\left(\delta_q^j\otimes I_k\right)\vec{x}=\mathbf{0}_k\right\}, & \text{for } x=x_e\\ \left\{\delta_q^j\in\Delta_q\,|\,\tilde{\lambda}^\top\mathbf{Q}^c\left(\delta_q^j\otimes I_k\right)\vec{x}<0\right\}, & \text{for } x\in\mathscr{D}_k\backslash\{x_e\}\end{cases} \tag{37}$$

where $\tilde{\lambda}\in\mathcal{R}^k$ is obtained by applying Lemma 4 to the CT-MC characterized by the TRM

$$\tilde{\mathbf{Q}}:=\frac{1}{q}\sum_{j=1}^{q}\mathbf{Q}^c\left(\delta_q^j\otimes I_k\right). \tag{38}$$

The admissible control sets $\mathcal{H}_x, x \in \mathscr{D}_k$, enable the construction of a state feedback controller $u(t) = \mathcal{K}(x(t))$ as follows.

Theorem 5: Given target state $x_e \in \mathscr{D}_k$, CT-CMC $\{\mathbf{Q}^c\}$ is asymptotically x_e-stablizable in distribution if and only if $\mathcal{H}_i \neq \emptyset$ holds for all $i \in \mathscr{D}_k$.

Proof: (Sufficiency.) The sufficiency is verified by constructing a feasible state feedback controller (35).

Define a constant vector $\hat{\lambda}$ as

$$\hat{\lambda}_i = \begin{cases} 0, & i = x_e \\ \tilde{\lambda}_i, & i \in \mathscr{D}_k \backslash \{x_e\}. \end{cases}$$

Then, the constant feedback gain matrix $K \in \mathcal{L}_{q\times k}$ is constructed as

$$\mathrm{Col}_i(K) \in \mathcal{H}_i,\ i \in \mathscr{D}_k.$$

The closed-loop system under (35) satisfies conditions (36a) and (36b) since $\hat{\lambda}_{x_e} = 0$ and $\hat{\lambda}_i > 0$ for all $i \in \mathscr{D}_k \backslash \{x_e\}$. Then, condition (36c) follows from the definition of $\mathcal{H}_{x_e}$ owing to

$$\hat{\lambda}^\top \mathbf{Q}^c (K \otimes I_k) \mathbf{M}_k \vec{x}_e = \hat{\lambda}^\top \mathbf{Q}^c (K \otimes I_k) \left(\delta_q^j \otimes I_k\right) \vec{x}_e = 0$$

for all $\delta_q^j \in \mathcal{H}_{x_e}$. Besides, condition (36d) is satisfied for $x \in \mathscr{D}_k \backslash \{x_e\}$ because of

$$\hat{\lambda}^\top \mathbf{Q}^c (K \otimes I_k) \mathbf{M}_k \vec{x} = \hat{\lambda}^\top \mathbf{Q}^c \left(\delta_q^j \otimes I_k\right) \vec{x} < 0$$

for all $\delta_q^j \in \mathcal{H}_x$ with $x \neq x_e$.

(Necessity.) To prove the necessity, we start by assuming that CT-CMC $\{\mathbf{Q}^c\}$ is asymptotically x_e-stablizable in distribution. First, the condition $\mathcal{H}_{x_e} \neq \emptyset$ must be satisfied; otherwise x_e cannot be the steady state of CT-CMC $\{\mathbf{Q}^c\}$.

Next, we prove that the stabilizability of CT-CMC $\{\mathbf{Q}^c\}$ implies the existence of a solution $\tilde{\lambda}$ such that $\mathcal{H}_x \neq \emptyset$ for all $x \in \mathscr{D}_k \backslash \{x_e\}$. The stabilizability of CT-CMC $\{\mathbf{Q}^c\}$ indicates that the state transition graph contains a spanning in-tree [17]. Moreover, the corresponding CT-MC $\{\tilde{\mathbf{Q}}\}$ inherits this property, ensuring the existence of $\tilde{\lambda}$ according to [19]. Through linearity, the condition

$$\tilde{\lambda}^\top \left(\frac{1}{q} \sum_{j=1}^{q} \mathbf{Q}^c (\delta_q^j \otimes I_k) \right) \vec{x} < 0$$

implies the existence of a parameter $j' \in \mathscr{D}_q$ such that

$$\tilde{\lambda}^\top \mathbf{Q}^c (\delta_q^{j'} \otimes I_k) \vec{x} < 0.$$

Therefore, set $\mathcal{H}_x$ is nonempty. ■

Theorem 5 provides a constructive approach for designing state feedback controllers to x_e-stabilize CT-CMC $\{\mathbf{Q}^c\}$. Specifically, the feasible state feedback controller (35) can be implemented by selecting

$$\mathrm{Col}_i(K) \in \mathcal{H}_i,\ i \in \mathscr{D}_k.$$

We then extend this result to the set stabilization of CT-CMC $\{\mathbf{Q}^c\}$, beginning with the definition of control invariant subsets in state set $\mathscr{D}_k$.

Definition 5 (See [17]): A subset $S \subseteq \mathscr{D}_k$ is said to be a control invariant subset of CT-CMC $\{\mathbf{Q}^c\}$ if, for any $j \in S$, there exists an integer $k_j \in \mathscr{D}_q$ such that

$$\sum_{i \in S} \left[\mathbf{Q}^c \left(\delta_q^{k_j} \otimes I_k \right) \right]_{i,j} = 0.$$

For any state subset $S \subseteq \mathscr{D}_k$, its largest control invariant subset $\mathcal{I}^*(S) \subseteq S$ can be computed via [17, Proposition 4]. Then, we define indicator matrix $\mathbb{I}_S \in \mathcal{R}^{k\times|S|}$ as follows:

$$\mathrm{Col}_j(\mathbb{I}_S) = \begin{cases} \delta_k^j, & j \in \mathcal{I}^*(S) \\ \delta_k^0, & j \in \mathscr{D}_k \backslash \mathcal{I}^*(S). \end{cases}$$

Note that CT-CMC $\{\mathbf{Q}^c\}$ achieves the stabilization to the given state set S if and only if it achieves stabilization to the largest control invariant subset $\mathcal{I}^*(S)$ of S. Therefore, regarding to CT-CMC $\{\mathbf{Q}^c\}$, we can construct the admissible control sets as follows:

$$\mathcal{H}_x^c := \begin{cases} \left\{ \delta_q^j \in \Delta_q \,|\, \mathbb{I}_S^\top \mathbf{Q}^c \left(\delta_q^j \otimes I_k\right) \vec{x} = 0 \right\}, \\ \qquad \text{for } x \in \mathcal{I}^*(S) \\ \left\{ \delta_q^j \in \Delta_q \,|\, \left(\tilde{\lambda}^c\right)^\top \mathbf{Q}^c \left(\delta_q^j \otimes I_k\right) \vec{x} < 0 \right\}, \\ \qquad \text{for } x \in \mathscr{D}_k \backslash \mathcal{I}^*(S) \end{cases} \quad (39)$$

where $\tilde{\lambda}^c \in \mathcal{R}^k$ is obtained by applying Lemma 5 to CT-MC $\{\tilde{\mathbf{Q}}\}$, whose TRM $\tilde{\mathbf{Q}}$ is calculated by (38).

Theorem 6: Given $S \subseteq \mathscr{D}_k$, CT-CMC $\{\mathbf{Q}^c\}$ is asymptotically S-stablizable in distribution if and only if $\mathcal{H}_i^c \neq \emptyset$ holds for all $i \in \mathscr{D}_k$. The feasible state feedback controller is designed as (35) with

$$\mathrm{Col}_i(K) \in \mathcal{H}_i^c,\ i \in \mathscr{D}_k.$$

Proof: The proof of this theorem follows a similar approach to the proof of Theorem 5, with

$$\hat{\lambda}_i = \begin{cases} 0, & i \in \mathcal{I}^*(S) \\ \tilde{\lambda}_i^c, & i \in \mathscr{D}_k \backslash \mathcal{I}^*(S). \end{cases}$$

Thus, the detailed proof process is omitted here. ■

In a broader sense, this Lyapunov-based feedback control design approach can be extended to other set-stabilization-related issues of CT-CMNs and CT-CMCs. For example, leader-follower CT-CMNs can be transformed into a compact CT-CMC using Theorem 2, and thus, their synchronization problem can be converted into a set stabilization problem subject to the corresponding CT-CMC, which can be solved through Theorem 6.

In comparison with [17], the feedback controllers designed in this work rely on the construction of a control Lyapunov function. Once the Lyapunov function is established, the feedback controllers can be readily derived by following Theorems 5 and 6. As indicated in [19], the construction of Lyapunov functions can be achieved through linear programming, which exhibits polynomial complexity with respect to the cardinality of the state spaces of the considered chains. Therefore, the proposed controller design method is computationally efficient.

V. Applications to Leader-Follower Synchronization and Output Regulation

In this section, we explore the applicability of the proposed methods to the synchronization of leader-follower CT-MNs and the output regulation of CT-CMNs.

A. Synchronization of Leader-Follower CT-MNs

The leader-follower CT-MN can be regarded as a special case of CT-MNs, which comprise a leader CT-MC and a follower CT-MC. Regarding the leader CT-MC, its state is denoted by $m(t) \in$

$\mathscr{D}_k$, and its dynamics is governed by TPM $P_m(t) \in \mathcal{P}_{k\times k}$, given as

$$[P_m(t)]_{i,j} := \Pr\{m(\theta+t)=i \,|\, m(\theta)=j\},\ i,j\in\mathscr{D}_k$$

with $\lim_{t\to 0^+} P_m(t) = P_m(0) = I_k$. Then, the TRM of the leader CT-MC is derived as

$$Q_m = \lim_{t\to 0^+} \frac{P_m(t)-I_k}{t}.$$

It satisfies the Kolmogorov equation

$$\dot{P}_m(t) = Q_m P_m(t)$$

with solution $P_m(t) = e^{Q_m t}$.

Next, concerning the follower CT-MC, its state trajectory, denoted by $s(t)\in\mathscr{D}_k$, is governed by both the leader CT-MC and itself. Define its TPM as

$$P_s(t) := [P_s^1(t)\, P_s^2(t) \cdots P_s^k(t)] \in \mathcal{P}_{k\times k^2}$$

with, for each $r\in\mathscr{D}_k$,

$$[P_s^r(t)]_{i,j} := \Pr\{s(\theta+t)=i \,|\, s(\theta)=j, m(\theta)=r\}.$$

It satisfies $\lim_{t\to 0^+} P_s^r(t) = P_s^r(0) = I_k$ for all $r\in\mathscr{D}_k$. Then, the TRM of the follower CT-MC subject to each $m(\theta)=r$ is derived as

$$Q_s^r := \lim_{t\to 0^+} \dot{P}_s^r(t).$$

For each $r\in\mathscr{D}_k$, it also satisfies

$$\dot{P}_s^r(t) = Q_s^r P_s^r(t)$$

with solution $P_s^r(t) = e^{Q_s^r t}$. Then, we further denote

$$Q_s := \left[Q_z^1, Q_z^2, \ldots, Q_z^k\right] \in \mathcal{R}_{k\times k^2}.$$

Let $\vec{m}(t) := \delta_k^{m(t)} \in \Delta_k$ and $\vec{s}(t) := \delta_k^{s(t)} \in \Delta_k$. The dynamics of the leader-follower CT-MN can be modeled as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{m}(t)\} = Q_m \mathbb{E}\{\vec{m}(t)\} \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{s}(t)\} = Q_s \mathbb{E}\{(\vec{m}(t)\otimes I_k)\,\vec{s}(t)\} \\ \mathbb{E}\{\vec{m}(0)\} =: \mathbf{p}_m(0) \in \mathcal{P}^k \\ \mathbb{E}\{\vec{s}(0)\} =: \mathbf{p}_s(0) \in \mathcal{P}^k. \end{cases} \tag{40}$$

Definition 6: Leader-follower CT-MC (40) is said to be asymptotically synchronous in distribution if

$$\lim_{t\to\infty} \Pr\{m(t)=s(t)\} = 1 \tag{41}$$

holds for any initial PDV $\mathbf{p}_m(0)\in\mathcal{P}^k$ and PDV $\mathbf{p}_s(0)\in\mathcal{P}^k$.

The synchronization problem of leader-follower CT-MC (40) can be reformulated using the augmented state variable

$$\vec{\xi}(t) = (\vec{m}(t)\otimes I_k)\,\vec{s}(t)$$

which transforms the original problem into the stabilization problem of a CT-MC with the target state set as follows:

$$\mathcal{S}_k := \left\{\delta_k^1\otimes\delta_k^1, \ldots, \delta_k^k\otimes\delta_k^k\right\}.$$

However, the leader-follower synchronization problem of CT-MC (40) presents significant technical challenges in the traditional framework [35] due to the difficulty in characterizing the expectation

$$\mathbb{E}\{(\vec{m}(t)\otimes I_k)\,\vec{s}(t)\}.$$

To overcome this limitation, we introduce the following conditional independence assumption: the random variables $m(t+\theta)$ and $s(t+\theta)$ are conditionally independent on $m(\theta)$ and $s(\theta)$ for all $t,\theta\geq 0$, i.e.,

$$\begin{aligned}&\Pr\{m(t+\theta), s(t+\theta) \mid m(\theta), s(\theta)\} = \\ &\Pr\{m(t+\theta) \mid m(\theta), s(\theta)\}\cdot\Pr\{s(t+\theta) \mid m(\theta), s(\theta)\}.\end{aligned}$$

Theorem 7: Leader-follower CT-MC (40) achieves asymptotical synchronization in distribution if and only if there exists a vector $\lambda\in\mathcal{R}_{k^2}$ satisfying

$$\lambda^\top\vec{\xi} \geq 0, \forall\vec{\xi}\in\Delta_{k^2} \tag{42a}$$

$$\lambda^\top\mathbf{Q}_{\text{ms}}\vec{\xi} \leq 0, \forall\vec{\xi}\in\mathcal{S}_k \tag{42b}$$

$$\lambda^\top\mathbf{Q}_{\text{ms}}\vec{\xi} < 0, \forall\vec{\xi}\in\Delta_{k^2}\backslash\mathcal{S}_k \tag{42c}$$

where

$$\mathbf{Q}_{\text{ms}} := Q_m\otimes I_k + \left(I_k\otimes\mathbf{1}_k^\top\right) * Q_s.$$

Proof: By Theorem 1, leader-follower CT-MC (40) under conditional independence admits the dynamics representation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left\{\vec{\xi}(t)\right\} = \mathbf{Q}_{\text{ms}}\mathbb{E}\left\{\vec{\xi}(t)\right\} \\ \mathbb{E}\left\{\vec{\xi}(0)\right\} = \mathbf{p}_m(0)\otimes\mathbf{p}_s(0). \end{cases} \tag{43}$$

Based on this, the proof can then be completed by directly applying Lemma 5 to (43). ■

B. Output Regulation of CT-CMNs

In this subsection, we investigate the output regulation problem for CT-CMNs, extending previous work on DT-CMNs [23]. The continuous-time case remains challenging due to the absence of coupling conditions.

Consider a CT-CMC with state variable $x(t)\in\mathscr{D}_{k_1}$, control variable $u(t)\in\mathscr{D}_q$, and output variable $y_1(t)\in\mathscr{D}_\eta$. Its dynamics is governed by TRM $Q^c := [Q_1^c, Q_2^c, \ldots, Q_q^c] \in \mathcal{R}_{k_1\times qk_1}$ with $Q_r^c\in\mathcal{R}_{k_1\times k_1}$, $r\in\mathscr{D}_q$. The PDV evolves according to the following ordinary differential equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{x}(t)\} = Q^c\mathbb{E}\{(\vec{u}(t)\otimes I_{k_1})\,\vec{x}(t)\} \\ y_1(t) = h_1(x(t)) \\ \mathbb{E}\{\vec{x}(0)\} =: \mathbf{p}_0^x \in \mathcal{P}^{k_1} \end{cases} \tag{44}$$

where $h_1 : \mathscr{D}_{k_1}\to\mathscr{D}_\eta$.

The reference system is a CT-MC with state variable $z(t)\in\mathscr{D}_{k_2}$ and output variable $y_2(t)\in\mathscr{D}_\eta$. Its dynamics is governed by TRM $Q\in\mathcal{R}_{k_2\times k_2}$ and obeys the following ordinary differential equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\vec{z}(t)\} = Q\mathbb{E}\{\vec{z}(t)\} \\ y_2(t) = h_2(z(t)) \\ \mathbb{E}\{\vec{z}(0)\} =: \mathbf{p}_0^z \in \mathcal{P}^{k_2} \end{cases} \tag{45}$$

where $h_2 : \mathscr{D}_{k_2}\to\mathscr{D}_\eta$.

Definition 7: The output regulation problem for system (44) and reference (45) is solved if there exists a state feedback controller

$$\vec{u}(t) = K\left(\vec{x}(t)\otimes I_{k_2}\right)\vec{z}(t) \tag{46}$$

such that the output synchronization condition

$$\lim_{t\to\infty}\Pr\{y_1(t)=y_2(t)\} = 1 \tag{47}$$

holds for all initial PDV $\mathbf{p}_0^x \in \mathcal{P}^{k_1}$ and PDV $\mathbf{p}_0^z \in \mathcal{P}^{k_2}$.

Assuming the conditional independence of $x(t)$ and $z(t)$, we define the augmented synchronization variable

$$\vec{\xi}(t) := (\vec{x}(t) \otimes I_{k_2})\, \vec{z}(t).$$

Based on this assumption, one can transform the output regulation problem into an asymptotic $\mathcal{O}$-stabilization problem in distribution for the CT-CMC with state $\vec{\xi}(t)$, where the synchronization set is

$$\mathcal{O}_{k_1k_2} := \Big\{(i,j) \in \mathscr{D}_{k_1} \times \mathscr{D}_{k_2} \mid (H_1 \otimes H_2)(\delta_{k_1}^i \otimes I_{k_2})\delta_{k_2}^j = \mathbf{M}_\eta \delta_\eta^s \,\forall s \in \mathscr{D}_\eta\Big\} \tag{48}$$

where H_1 and H_2 are structure matrices of h_1 and h_2, respectively. This formulation enables a direct application of Theorems 2 and 6 to the output regulation problem.

Theorem 8: The output regulation of system (44) and reference (45) can be achieved under state feedback control (46) if and only if there exists a vector $\lambda \in \mathcal{R}^{k_1k_2}$ satisfying

$$\begin{aligned}&\bar{\lambda}^\top \vec{\xi} \geq 0, \forall \vec{\xi} \in \mathscr{D}_{k_1k_2}\\ &\bar{\lambda}^\top \bar{\mathbf{Q}}^c \vec{\xi} \leq 0, \forall \vec{\xi} \in \mathcal{I}^*(\mathcal{O}_{k_1k_2})\\ &\bar{\lambda}^\top \bar{\mathbf{Q}}^c \vec{\xi} < 0, \forall \vec{\xi} \in \mathscr{D}_{k_1k_2} \backslash \mathcal{I}^*(\mathcal{O}_{k_1k_2})\end{aligned}$$

where

$$\begin{aligned}\bar{\mathbf{Q}}^c :=& \left[Q^c(I_{qk_1} \otimes \mathbf{1}_{k_2}^\top) K \mathbf{M}_{k_1k_2}\right] * \left(\mathbf{1}_{k_1}^\top \otimes I_{k_2}\right)\\ &+ \left[(I_{qk_1} \otimes \mathbf{1}_{k_2}^\top) K \mathbf{M}_{k_1k_2}\right] * \left[Q(\mathbf{1}_{k_1}^\top \otimes I_{k_2})\right] \in \mathcal{R}_{k_1k_2 \times k_1k_2}.\end{aligned}$$

After obtaining λ through Theorem 8, feasible state feedback controllers can be designed as

$$\vec{u}(t) = \bar{K}\vec{\xi}(t) \tag{49}$$

with

$$\mathrm{Col}_\xi(\bar{K}) \in \bar{\mathcal{H}}_\xi^c,\ \xi \in \mathscr{D}_k.$$

Here, $\bar{\mathcal{H}}_\xi^c$ for each $x \in \mathscr{D}_k$ is constructed according to (39) subject to the TRM $\bar{\mathbf{Q}}^c$ as follows:

$$\bar{\mathcal{H}}_\xi^c = \begin{cases}\left\{\delta_q^r \in \Delta_q \mid \mathbb{I}_{\mathcal{O}_{k_1k_2}}^\top \bar{\mathbf{Q}}^c \left(\delta_q^r \otimes I_{k_1k_2}\right)\vec{\xi} = 0\right\}, & \text{for } \xi \in \mathcal{I}^*(\mathcal{O}_{k_1k_2})\\ \left\{\delta_q^r \in \Delta_q \mid \bar{\lambda}^\top \bar{\mathbf{Q}}^c \left(\delta_q^r \otimes I_{k_1k_2}\right)\vec{\xi} < 0\right\}, & \text{for } \xi \in \mathscr{D}_{k_1k_2} \backslash \mathcal{I}^*(\mathcal{O}_{k_1k_2}).\end{cases}$$

VI. Simulating Results

This section presents the following three practical applications of our theoretical framework:

1) a resource allocation problem in networked systems;
2) a synchronization analysis for tandem queueing systems;
3) a numerical case study of CT-CMNs.

Through these diverse examples, we can demonstrate the applicability of our method for analyzing various types of CT-MNs and CT-CMNs, particularly in addressing synchronization and output regulation challenges.

A. Resource Sharing Problem

In this subsection, we examine the resource-sharing problem introduced in [31, Sec. 3.3.1], where n distinct processes share a common resource. Each process operates in two states: a *sleeping state* (inactive) and a *resource-using state* (active). The system enforces a strict concurrency limit r $(1 \leq r \leq n)$, meaning that any process attempting to transition from the sleeping state to the resource-using state will be immediately blocked and forced back to the sleeping state if r processes are already actively using the resource.

This model captures two well-known cases: 1) when $r = 1$, it simplifies to the classic *mutual exclusion* problem; and 2) when $r = n$, all processes operate independently without contention, as the resource can accommodate every process simultaneously. To formalize the process dynamics, we define for each process $i \in \mathscr{D}_n$ that λ_i is the *activation rate* at which process i initiates a resource request from the sleeping state, and μ_i is the *release rate* at which process i relinquishes the resource upon completing its task.

In what follows, we model each process $i \in \mathscr{D}$ as a two-state CT-MC: Sleeping and Using. Let $x_i(t) \in \{\text{Sleeping}, \text{Using}\}$ be the state of process i at time t, and denote by $x := (x_1, x_2, \ldots, x_n)^\top \in \{\text{Sleeping}, \text{Using}\}^n$ the system state. The congestion control is governed by a Boolean-valued resource availability function

$$f(x) = \begin{cases}1, & \sum_{j=1}^n |\{x_j = \text{Using}\}| < r\\ 0, & \sum_{j=1}^n |\{x_j = \text{Using}\}| \geq r.\end{cases} \tag{50}$$

Here, $f(x) = 1$ indicates resource availability (i.e., fewer than r processes in Using state), while $f(x) = 0$ denotes the resource congestion.

The transition dynamics for process $i \in \mathscr{D}_n$ are specified as follows:

1) process i transfers from Sleeping to Using at rate $\lambda_i f(x)$;
2) process i remains Sleeping at rate $-\lambda_i f(x)$;
3) process i transfers from Using to Sleeping at rate μ_i;
4) process i remains Using at rate $-\mu_i$.

This yields the state-dependent TRM $Q_i(x)$ for process $i \in \mathscr{D}_n$ as

$$Q_i(x) = \begin{bmatrix}-\lambda_i f(x) & \mu_i\\ \lambda_i f(x) & -\mu_i\end{bmatrix}, \forall x \in \{\text{Sleeping}, \text{Using}\}^n. \tag{51}$$

Thus, the resource-sharing system can be formally modeled as a CT-MN composed of n interacting CT-MCs, where each component process $i \in \mathscr{D}_n$ evolves according to its state-dependent TRM Q_i. This networked structure captures both the individual process dynamics and their coupling through the shared resource constraint.

To validate our theoretical framework, we instantiate the model with specific parameters: $n = 3$ processes sharing $r = 3$ resource units, with homogeneous transition rates $\lambda_i = 1$ and $\mu_i = 2$ for all $i \in \mathscr{D}_3$. Fig. 2 depicts the network structure of the three-node resource sharing model. Adopting the state encoding Using $\equiv \delta_2^1$ and Sleeping $\equiv \delta_2^2$, we derive the state-dependent TRM as

$$Q_1 = \begin{bmatrix}0 & -1 & -1 & -1 & 2 & 2 & 2 & 2\\ 0 & 1 & 1 & 1 & -2 & -2 & -2 & -2\end{bmatrix} \in \mathcal{R}_{2\times 8}$$

$$Q_2 = \begin{bmatrix}0 & -1 & 2 & 2 & -1 & -1 & 2 & 2\\ 0 & 1 & -2 & -2 & 1 & 1 & -2 & -2\end{bmatrix} \in \mathcal{R}_{2\times 8}$$

$$Q_3 = \begin{bmatrix}0 & 2 & -1 & 2 & -1 & 2 & -1 & 2\\ 0 & -2 & 1 & -2 & 1 & -2 & 1 & -2\end{bmatrix} \in \mathcal{R}_{2\times 8}.$$

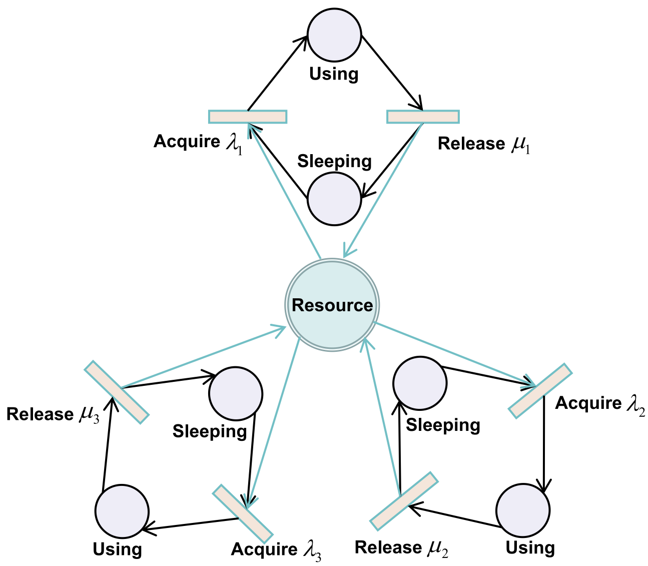

Fig. 2. Graphical illustration of the resource sharing model.

By Theorem 1, the resource-sharing CT-MN admits an equivalent representation as a single CT-MC with the TRM as follows:

$$Q=\begin{bmatrix}0 & 2 & 2 & 0 & 2 & 0 & 0 & 0\\ 0 & -4 & 0 & 2 & 0 & 2 & 0 & 0\\ 0 & 0 & -4 & 2 & 0 & 0 & 2 & 0\\ 0 & 1 & 1 & -5 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & -4 & 2 & 2 & 0\\ 0 & 1 & 0 & 0 & 1 & -5 & 0 & 2\\ 0 & 0 & 1 & 0 & 1 & 0 & -5 & 2\\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & -6\end{bmatrix}\in\mathcal{R}_{8\times 8}.$$

Next, we consider the synchronization problem of this resource-sharing system, and thus, assign the target state $\vec{x}_e=\delta_8^1$, which corresponds to the case of $x_e=(\text{Using,Using,Using})$. By Lemma 4, the equivalent CT-MC admits a Lyapunov function vector

$$\lambda=[0,2,5,4,2,3,4,4]^\top.$$

Therefore, one can yield that, for any initial distribution, this resource-sharing system is stable at state $(\text{Using,Using,Using})^\top$, which is also clear from the trajectories of probability $\Pr\{x(t)=x_e\}$ depicted in Fig. 3.

B. Tandem-Queue Problem

Consider the tandem-queue model introduced in [32], which consists of two cascaded queues. In this model, customers arrive at the first queue with rate λ and are served at rate μ_1. Upon service completion, they proceed to the second queue with probability p or exit the system with probability $1-p$. The system architecture is illustrated in Fig. 4.

The model defines the following two possible states for customers: 1) *entering the system*, denoted by 1, and 2) *getting the service*, denoted by 2. This structure forms a leader-follower CT-MN. The TRM of the leader CT-MC, Q_m, is given by

$$Q_m=\begin{bmatrix}-\lambda & \mu_1(1-p)\\ \lambda & -\mu_1(1-p)\end{bmatrix}\in\mathcal{R}_{2\times 2}$$

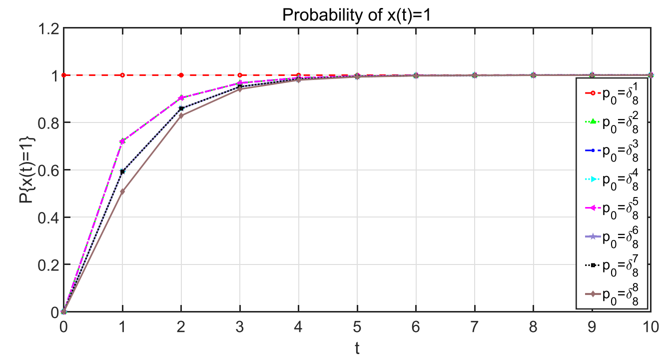

Fig. 3. Trajectories of probability $\Pr\{x(t)=x_e\}$ with initial PDV $\mathbf{p}_0\in\{\delta_8^1,\delta_8^2,\ldots,\delta_8^8\}$.

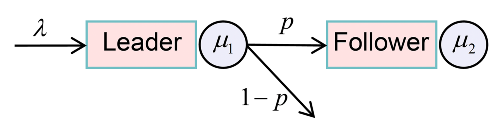

Fig. 4. Graphical illustration of the tandem-queue model.

while the TRM of the follower CT-MC, Q_s, is defined as

$$Q_s=\begin{bmatrix}0 & \mu_2 & 0 & \mu_1 p\\ 0 & -\mu_2 & 0 & -\mu_1 p\end{bmatrix}\in\mathcal{R}_{2\times 4}.$$

By Theorem 7, the TRM $\mathbf{Q}_{\text{ms}}$ of this leader-follower CT-MN can be calculated as

$$\mathbf{Q}_{\text{ms}}=Q_m\otimes I_2+\left(I_2\otimes \mathbf{1}_2^\top\right)*Q_s.$$

For the parameter values $\lambda=13, p=0.7, \mu_1=20, \mu_2=30$, we obtain

$$\mathbf{Q}_{\text{ms}}=\begin{bmatrix}-13 & 30 & 6 & 0\\ 0 & -43 & 0 & 6\\ 13 & 0 & -6 & 14\\ 0 & 13 & 0 & -20\end{bmatrix}.$$

In the synchronization problem, the set of convergence states is given by

$$\mathcal{S}_e=\left\{(1,1)^\top,(2,2)^\top\right\}.$$

According to Theorem 7, there does not exist a suitable vector λ for this configuration. However, if we instead consider the modified set as

$$\Lambda=\left\{(1,1)^\top,(2,1)^\top\right\}$$

there exists a solution vector λ satisfying the conditions of Theorem 7, specifically

$$\lambda=[0,10,0,8]^\top.$$

Consequently, although this leader-follower CT-MN does not achieve complete synchronization, it exhibits asymptotic synchronization toward the state $(1,1)^\top$ with a stationary probability of 0.3, as demonstrated in Fig. 5. Furthermore, this system is asymptotically Λ-stable in distribution, as defined in Definition 1 and indicated in Fig. 6.

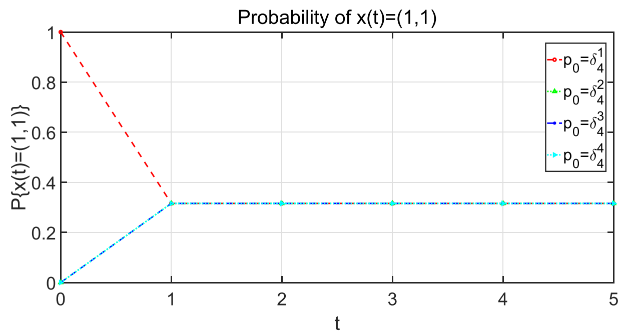

Fig. 5. Trajectories of $\Pr\{x(t)=(1,1)\}$ with initial PDV $\mathbf{p}_0 \in \Delta_4$.

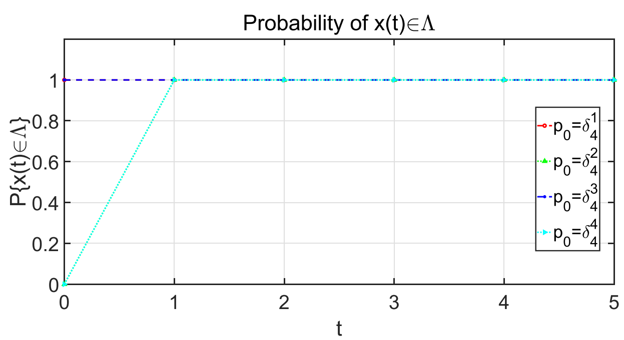

Fig. 6. Trajectories of $\Pr\{x(t) \in \Lambda\}$ with initial PDV $\mathbf{p}_0 \in \Delta_4$.

C. Output Regulation of Numerical CT-CMNs

In this subsection, we present a numerical simulation for the output regulation problem in CT-CMN (44). The TRM $Q^c = [Q_1^c\, Q_2^c]$ is given as

$$Q_1^c = \begin{bmatrix} -3 & 2 & -1 & 4 \\ 3 & -2 & 1 & -4 \end{bmatrix}, \; Q_2^c = \begin{bmatrix} -5 & 0 \\ 5 & 0 \end{bmatrix}$$

and the feedback gain matrices of h_1 and h_2 are designed as

$$H_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This system admits an equivalent representation in the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left\{\vec{\xi}(t)\right\} = \bar{\mathbf{Q}}^c \left(\vec{u}(t) \otimes I_4\right) \mathbb{E}\left\{\vec{\xi}(t)\right\} \\ \vec{y}(t) = H\vec{\xi}(t) \end{cases}$$

where $\vec{\xi} = \vec{x} \otimes \vec{z}$ represents the composite state vector, $H = H_1 \otimes H_2 = \delta_4[2,1,2,1]$ is the output matrix, and

$$\bar{\mathbf{Q}}^c = \begin{bmatrix} -8 & 0 & 2 & 0 & -6 & 0 & 4 & 0 \\ 5 & -3 & 0 & 2 & 5 & -1 & 0 & 4 \\ 3 & 0 & -7 & 0 & 1 & 0 & -9 & 0 \\ 0 & 3 & 5 & -2 & 0 & 1 & 5 & -4 \end{bmatrix}.$$

According to (48), the desired synchronization set $\mathcal{O}_4$ is computed as

$$\mathcal{O}_4 = \{(1,2),(2,2)\} \sim \{2,4\} \subseteq \mathscr{D}_4.$$

To achieve our output regulation objective, we begin by computing the invariant set $\mathcal{I}^*(\mathcal{O}_4)$ for the desired synchronization set $\mathcal{O}_4$. Our analysis reveals that $\mathcal{I}^*(\mathcal{O}_4) = \mathcal{O}_4$, indicating that

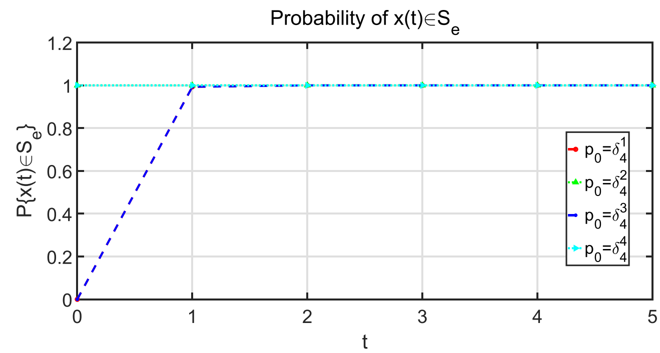

Fig. 7. Trajectories of $\Pr\{x(t) \in S_e\}$ with initial PDV $\mathbf{p}_0 \in \Delta_4$.

$\mathcal{O}_4$ is itself invariant. For each state $\xi \in \mathcal{I}^*(\mathcal{O}_4)$, we define the admissible control set

$$\bar{\mathcal{H}}_\xi^c = \left\{ \delta_2^r \in \Delta_2 \,|\, \mathbb{I}_{\mathcal{O}_4}^\top \bar{\mathbf{Q}}^c(\delta_2^r \otimes I_4)\vec{\xi} = 0 \right\} = \left\{\delta_2^1, \delta_2^2\right\}$$

By considering the averaged dynamics matrix $\mathcal{Q} = \frac{Q_1^c + Q_2^c}{2} \in \mathcal{R}_{4\times 4}$, we construct a Lyapunov function $V(\vec{\xi}) = \bar{\lambda}^\top \vec{\xi}$ for CT-MC $\{\mathcal{Q}\}$ with the coefficient vector:

$$\bar{\lambda} = [2,0,3,0]^\top.$$

For transient states $\xi \in \{1,3\}$, we further define the admissible control set

$$\bar{\mathcal{H}}_\xi^c := \left\{ \delta_2^r \in \Delta_2 \,|\, \bar{\lambda}^\top \bar{\mathbf{Q}}^c(\delta_2^r \otimes I_4)\vec{\xi} < 0 \right\} = \left\{\delta_2^1, \delta_2^2\right\}.$$

Consequently, output regulation of this CT-CMN can be achieved through state feedback controller (49) with $\bar{K}$ as

$$\mathrm{Col}_i(\bar{K}) \in \left\{\delta_2^1, \delta_2^2\right\}, \; i \in \mathscr{D}_4.$$

As a concrete example, consider the feedback gain matrix

$$\bar{K} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Fig. 7 demonstrates the system trajectories under this controller, clearly showing the convergence to the desired output regulation objective from various initial conditions.

VII. Conclusion

In this article, we have established a state-space representation for CT-MNs and CT-CMNs by introducing the coupling Markov property and conditional independence property. Our framework has not only clarifies the relationship between the state-space representation of CT-MNs and prior work [25], but also unifies these perspectives within a generalized formalism. From a Lyapunov-based control perspective, we have designed state feedback controllers through solutions to linear programming problems, providing an energy-decreasing interpretation of system stabilization. The proposed methodology has resolve two previously ill-posed problems: synchronization in leader-follower CT-MNs and output regulation for CT-CMNs. While these applications serve as our primary focus, the developed approach has also naturally extended to broader set stability and set stabilization problems.

Acknowledgment

The first author would like to thank Mr. S. Zhu from the Department of Electrical Engineering at the City University of Hong Kong for his valuable ideas and suggestions.

References

[1] W.-K. Ching and M. K. Ng, *Markov Chains*. New York, NY, USA: Springer, 2006.
[2] L. Lin, J. Cao, S. Zhu, and P. Shi, "Synchronization analysis for stochastic networks through finite fields," *IEEE Trans. Autom. Control*, vol. 67, no. 2, pp. 1016–1022, Feb., 2022.
[3] S. A. Kauffman, "Metabolic stability and epigenesis in randomly constructed genetic nets," *J. Theor. Biol.*, vol. 22, no. 3, pp. 437–467, 1969.
[4] I. Shmulevich, E. R. Dougherty, and W. Zhang, "From Boolean to probabilistic Boolean networks as models of genetic regulatory networks," *Proc. IEEE*, vol. 90, no. 11, pp. 1778–1792, Nov., 2002.
[5] S. Zhu, J. Lu, D. W. Ho, and J. Cao, "Minimal control nodes for strong structural observability of discrete-time iterative systems: Explicit formulas and polynomial-time algorithms," *IEEE Trans. Autom. Control*, vol. 69, no. 4, pp. 2158–2173, Apr., 2024.
[6] A. Fagiolini and A. Bicchi, "On the robust synthesis of logical consensus algorithms for distributed intrusion detection," *Automatica*, vol. 49, no. 8, pp. 2339–2350, 2013.
[7] D. Cheng and T. Liu, "From Boolean game to potential game," *Automatica*, vol. 96, pp. 51–60, 2018.
[8] G. Zhao, Y. Wang, and H. Li, "A matrix approach to the modeling and analysis of networked evolutionary games with time delays," *IEEE/CAA J. Automatica Sinica*, vol. 5, no. 4, pp. 818–826, Jul., 2018.
[9] C. Andrieu, N. De Freitas, A. Doucet, and M. I. Jordan, "An introduction to MCMC for machine learning," *Mach. Learn.*, vol. 50, pp. 5–43, 2003.
[10] S. P. Singh and R. S. Sutton, "Reinforcement learning with replacing eligibility traces," *Mach. Learn.*, vol. 22, no. 1, pp. 123–158, 1996.
[11] Y. Zhao and D. Cheng, "On controllability and stabilizability of probabilistic Boolean control networks," *Sci. China–Inf. Sci.*, vol. 57, 2014, Art. no. 012202.
[12] M. Meng, L. Liu, and G. Feng, "Stability and l_1 gain analysis of Boolean networks with Markovian jump parameters," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 4222–4228, Aug., 2017.
[13] Y. Guo, R. Zhou, Y. Wu, W. Gui, and C. Yang, "Stability and set stability in distribution of probabilistic Boolean networks," *IEEE Trans. Autom. Control*, vol. 64, no. 2, pp. 736–742, Feb., 2019.
[14] R. Zhou, Y. Guo, Y. Wu, and W. Gui, "Asymptotical feedback set stabilization of probabilistic Boolean control networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 11, pp. 4524–4537, Nov., 2020.
[15] S. Zhu, J. Lu, and D. W. C. Ho, "Finite-time stability of probabilistic logical networks: A topological sorting approach," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 67, no. 4, pp. 695–699, Apr., 2020.
[16] S. Zhu, J. Cao, L. Lin, L. Rutkowski, J. Lu, and G. Lu, "Observability and detectability of stochastic labeled graphs," *IEEE Trans. Autom. Control*, vol. 68, no. 12, pp. 7299–7311, Dec., 2023.
[17] Y. Guo, Z. Li, Y. Liu, and W. Gui, "Asymptotical stability and stabilization of continuous-time probabilistic logic networks," *IEEE Trans. Autom. Control*, vol. 67, no. 1, pp. 279–291, Jan., 2022.
[18] L. Lin et al., "Stabilization of continuous-time probabilistic logical networks under sampling dwell time constraints," *SIAM J. Control Optim.*, vol. 62, no. 2, pp. 1006–1033, 2024.
[19] S. Zhu, Y. Li, J. Cao, G. M. Dimirovski, and J. Lu, "Lyapunov-type criteria of absorbing continuous-time Markov chains," *IEEE Trans. Autom. Control*, vol. 69, no. 4, pp. 2422–2428, Apr., 2024.
[20] Z. Li, Y. Guo, and W. Gui, "Asymptotical stability of continuous-time probabilistic logic networks based on transition rate," *Sci. China Inf. Sci.*, vol. 66, no. 3, 2023, Art. no. 132201.
[21] Z. Li, Y. Guo, and W. Gui, "Asymptotical controllability of continuous-time probabilistic logic control systems," *IEEE Trans. Autom. Control*, vol. 69, no. 4, pp. 2769–2776, Apr., 2024.
[22] H. Li, Y. Wang, and L. Xie, "Output tracking control of Boolean control networks via state feedback: Constant reference signal case," *Automatica*, vol. 59, pp. 54–59, 2015.
[23] H. Li, L. Xie, and Y. Wang, "Output regulation of Boolean control networks," *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 2993–2998, Jun., 2017.
[24] Y. Guo, P. Wang, W. Gui, and C. Yang, "Set stability and set stabilization of Boolean control networks based on invariant subsets," *Automatica*, vol. 61, pp. 106–112, 2015.
[25] D. Cheng, H. Qi, and Z. Li, *Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach*. London, U.K.: Springer-Verlag, 2011.
[26] C. Li, X. Zhang, J.-e. Feng, and D. Cheng, "Transition analysis of stochastic logical control networks," *IEEE Trans. Autom. Control*, vol. 69, no. 2, pp. 1226–1233, Feb., 2024.
[27] L. Lin, J. Cao, J. Lam, L. Rutkowski, G. M. Dimirovski, and S. Zhu, "A bisimulation-based foundation for scale reductions of continuous-time Markov chains," *IEEE Trans. Autom. Control*, vol. 69, no. 9, pp. 5743–5758, Sep., 2024.
[28] W. J. Anderson, *Continuous-Time Markov Chains: An Applications-Oriented Approach*, Berlin, Germany: Springer, 2012.
[29] U. Nodelman, C. R. Shelton, and D. Koller, "Continuous-time Bayesian networks," in *Proc. 18th Conf. Uncertainty Artif. Intell.*, 2002, pp. 378–387.
[30] T. El-Hay, N. Friedman, D. Koller, and R. Kupferman, "Continuous-time Markov networks," in *Proc. 22nd Conf. Uncertainty Artif. Intell.*, 2006, pp. 155–164.
[31] P. Fernandes, B. Plateau, and W. J. Stewart, "Efficient descriptor-vector multiplications in stochastic automata networks," *J. ACM*, vol. 45, no. 3, pp. 381–414, 1998.
[32] M. Abadi et al., "Tensorflow: A system for large-scale machine learning," in *Proc. 12th USENIX Symp. Operating Syst. Des. Implementation*, 2016, pp. 265–283.
[33] G. Stoll, E. Viara, E. Barillot, and L. Calzone, "Continuous time Boolean modeling for biological signaling: Application of Gillespie algorithm," *BMC Syst. Biol.*, vol. 6, no. 1, pp. 1–18, 2012.
[34] B. Plateau, "On the stochastic structure of parallelism and synchronization models for distributed algorithms," in *Proc. ACM SIGMETRICS Conf. Meas. Model. Comput. Syst.*, 1985, pp. 147–154.
[35] R. Li, M. Yang, and T. Chu, "State feedback stabilization for probabilistic Boolean networks," *Automatica*, vol. 50, no. 4, pp. 1272–1278, 2014.

Changxi Li received the Ph.D. degree in control science and engineering from the Harbin Institute of Technology, Harbin, China, in 2020.

He held a Postdoctoral Research position with Shandong University, Jinan, China, from 2020 to 2022. Since 2022, he has been a Postdoctoral Researcher with the College of Engineering, Peking University, Beijing, China. His current research interests include stochastic processes, game-theoretical control, Boolean control networks, and distributed control of microgrids.

Dr. Li has been a Young Expert with Taishan Scholars of Shandong province since 2024.

Lin Lin received the Ph.D. degree in engineering from the Department of Mechanical Engineering, University of Hong Kong, Hong Kong, in 2024.

She held a Postdoctoral Fellow position with the Department of Electrical Engineering, City University of Hong Kong, Hong Kong. She has visiting positions in Australia, England, and Japan. She is currently a Postdoctoral Fellow with the University of Hong Kong. Her research interests include networked collective intelligence, logical networks, control theory, and reinforcement learning.

Dr. Lin was the recipient of the IETI PhD Fellowship Award, the Lotfi Zadeh Best Paper Award Finalist, and the Outstanding Master Degree Thesis Award from the Chinese Institute of Electronics.

Wenjun Xiong was born in Hubei Province, China. She received the M.Sc. degree in applied mathematics from the Department of Mathematics, Southeast University, Nanjing, China, in 2005, and the Ph.D. degree in mathematics from the Department of Mathematics, City University of Hong Kong, Hong Kong, in 2010.

She is currently a Professor with the School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu, China. Her research interests include iteration learning control, multiagent systems, complex networks, nonlinear dynamics and control, neural networks, and cooperative control of autonomous systems.

James Lam (Fellow, IEEE) received the B.Sc. (First Hons.) degree in mechanical engineering from the University of Manchester, Manchester, U.K., in 1983, and the M.Phil. degree in control and operational research and Ph.D. degree in control engineering from the University of Cambridge, Cambridge, U.K., in 1985 and 1988, respectively.

Prior to joining the University of Hong Kong, Pokfulam, Hong Kong, in 1993, where he is currently a Chair Professor of control engineering, he was a faculty member with the City University of Hong Kong and the University of Melbourne. His research interests include model reduction, robust synthesis, delay, singular systems, stochastic systems, multidimensional systems, positive systems, networked control systems, and vibration control.

Dr. Lam is a Croucher Scholar, Croucher Fellow, and Distinguished Visiting Fellow of the Royal Academy of Engineering, and a Cheung Kong Chair Professor. He is a Chartered Mathematician, Chartered Scientist, Chartered Engineer, and a Fellow of the Institute of Electrical and Electronic Engineers, the Institution of Engineering and Technology, the Institute of Mathematics and Its Applications, the Institution of Mechanical Engineers, and the Hong Kong Institution of Engineers. He is the Editor-in-Chief for *IET Control Theory and Applications*, *Journal of The Franklin Institute*, *Proc. IMechE Part I: Journal of Systems and Control Engineering*, *IET Journal of Engineering*, and *Franklin Open*, a Subject Editor for *Journal of Sound and Vibration*, an Editor for *Asian Journal of Control*, a Section Editor for a Consulting Editor for *International Journal of Systems Science*, and an Associate Editor for *Automatica* and *Multidimensional Systems and Signal Processing*. He is a Highly Cited Researcher in Engineering from 2014 to 2020, Cross-Fields in 2021, and Computer Science in 2015. He is a Member of Academia Europaea, and an Academician of the International Academy of Systems and Cybernetic Sciences.

Ka-Wai Kwok (Senior Member, IEEE) received the B.Eng. and M.Phil. degrees from the Department of Automation and Computer-aided Engineering (aka. Mechanical and Automation Engineering (MAE) currently), The Chinese University of Hong Kong (CUHK), Hong Kong, in 2003 and 2005, respectively, and the Ph.D. degree in computing from the Hamlyn Centre for Robotic Surgery, Department of Computing, Imperial College London, in 2012.

He was awarded the Croucher Foundation Fellowship 2013, which supported his research jointly supervised by advisors in The University of Georgia, and Brigham and Women's Hospital, Harvard Medical School. He served as a faculty with the Department of Mechanical Engineering, The University of Hong Kong (HKU), in 2014–2024. He is currently a Professor with the MAE, CUHK. He is the Principal Investigator of a research group for Interventional Robotic and Imaging Systems (IRIS), which has more than five inventions licensed/transferred from university to industry in support for their commercialization. He is also a cofounder and Director of Agilis Robotics Limited aiming at advancing the interventional endoscopy with small, flexible robotic instruments, and their intelligent control systems. His team successfully performed world-first transurethral robotic *en-bloc* resection of bladder tumours (ERBT) in live humans in 2024. He has participated in various designs of robotic devices/interfaces for endoscopy, and MRI-guided interventions. To date, he has coauthored more than 180 peer-reviewed articles with more than 80 clinical fellows and more than 160 scientists/engineers. His research interests include surgical robotics, intraoperative image processing, and their uses of control and intelligent systems.

Dr. Kwok was the recipient of ICRA Best Conference Paper Award in 2018 and IROS Toshio Fukuda Young Professional Award in 2020. He was also the recipient of many awards in his early career for robotics, e.g., the Early Career Awards 2015/16 offered by Research Grants Council (RGC) of Hong Kong, Actuators 2020 Young Investigator Award, HKU 2019–2020 Outstanding Young Researcher Award, HKU Young Innovator Award 2020, and HKU Research Output Prize 2021–22. His multidisciplinary work has been recognized with various (more than ten) awards in international conferences/journals, e.g., the largest flagship conferences of robotics, IEEE International Conference on Robotics and Automation (ICRA) and IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS).