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Reachable Set Estimation of Switched Systems with All Subsystems Unstable

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***Abstract*—Reachable set estimation of switched systems consisting entirely of unstable subsystems poses a challenge that remains unsolved, particularly when employing Lyapunov methods. In this work, a lemma for bounding the Lyapunov functions of switched continuous-time systems with dwell time confined to a bounded interval is proposed. For switched systems with all subsystems unstable, the main difficulty in estimating the reachable set is that conventional constant symmetric positive definite matrices in the quadratic Lyapunov functions (referred to as Lyapunov matrices in this work) will lead to infeasible conditions due to parameter restrictions. To obtain implementable reachable set estimation conditions, our novelty is constructing time-varying Lyapunov matrix functions to develop sufficient conditions for enclosing the reachable set of all unstable subsystems. The core idea is to use the derivatives of matrix functions over a finite time domain to compensate for restrictions on parameters. Furthermore, a linear interpolation approach is utilized to enable time-varying Lyapunov matrix functions. Superior conservatism reduction features are showcased by numerical simulation results.**

***Index Terms*—Reachable set estimation, switched systems, unstable subsystems, time-varying Lyapunov approach.**

I. Introduction

Switched systems in the control field refer to a class of hybrid dynamic systems that consist of continuous dynamics—subsystems represented by differential or difference equations, and discrete dynamics—discrete switching events arising from inherent changes or environmental influences [1], [2]. The switched system model exhibits great application potential in various complex practical processes. For instance, switched control of robots based on task requirements [3]. Wireless networks switch their communication topologies based on spatial or temporal variations [4], [5]. Manufacturing processes switch between failure and recovery status [6]. Owing to the vast applications of switching systems, many valuable theoretical results have been developed rapidly. In particular, stability and stabilization problems of switched systems have been investigated under various kinds of switching laws, including arbitrary switching [7], state-dependent switching [8], dwell time switching [9], average dwell time switching [10], persistent dwell time switching [11], and so on. By employing the well-known common Lyapunov approach and multiple Lyapunov approaches, most of these studies focused on stable and stabilizable subsystems. However, unstable modes should not be overlooked given practical scenarios of networked systems under recoverable attacks or faults possibly resulting in intermittent unstable dynamics, and unstable dynamics over mismatched intervals arising from asynchronous switching. A recognized feature of switched systems is that even with the presence of unstable/unstabilizable subsystems, the overall stability of the switched systems can be achieved through specific switching laws. The stability and stabilization results of switched systems with unstable subsystems have been reported in [12]–[14]. The main techniques for handling unstable dynamics are through selecting design parameters to compensate for the divergence of unstable subsystems to develop feasible conditions.

Moreover, practical systems are constantly affected by unexpected disturbances that may drive the evolution of a control system into undesired regions, and result in unsafe hazards. Evaluating the system evolution under disturbance effects motivates studies on the reachable set of dynamic systems. Reachable set refers to a set of all system states that can reach from the origin. Generally speaking, computing the exact reachable set is cost-consuming and impractical. Thus, reachable set estimation is widely investigated as an alternative strategy to find a bounded set to enclose the reachable set of systems under bounded disturbance input [15]. Reachable set estimation is closely related to practical applications such as constrained control [16], safety verification [17], and collision avoidance [18]. There are a lot of reachable set estimation results for various types of systems [19]–[22]. They usually use the Lyapunov approach to estimate the reachable set with ellipsoidal regions, leveraging the tractability offered by Lyapunov quadratic forms. The ellipsoidal estimates could provide an alternative method for interval estimation of dy-

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namic systems [23]–[25] by finding the smallest box enclosing the ellipsoids [26]. Regarding switched systems subject to bounded-peak disturbances, reachable set estimation was investigated in [27] for discrete-time switched systems under arbitrary switching by using a multiple Lyapunov approach. A work [28] estimated the reachable set of continuous-time switched systems under dwell time switching, which adopts a time-scheduled Lyapunov approach. Along the studies on reachable set estimation of switched systems under dwell time switching, different conditions were derived in [29] to bound the reachable set while allowing a bounded increase of Lyapunov functions at the switching instants. In [30], the reachable set of discrete-time switched systems under mode-dependent average dwell time switching was estimated using a discontinuous multiple Lyapunov approach. However, these reported reachable set estimation results for switched systems are predominantly limited to the case of stable subsystems. To the knowledge of the authors, the reachable set estimation for continuous-time switched systems with unstable subsystems has not been investigated.

Although there are stability results of switched systems with unstable subsystems, the reachable set estimation for such systems is not a trivial extension. Generally, stability analysis focuses on the steady-state performance of dynamic systems, while reachable set analysis concentrates on bounding the state trajectories. A notable challenge arises from the restrictions on parameters imposed by the reachable set estimation based on Lyapunov approaches, leading to infeasible conditions for unstable subsystems with any constant Lyapunov matrices. Therefore, existing time-scheduled Lyapunov functions containing constant Lyapunov matrices [12], [31], [32] are not applicable to our work. To address this challenging issue, inspired by [33], our work will explore novel time-varying matrix function constructions using lower and upper bounds of dwell times to establish feasible reachable set estimation conditions for switched systems with all subsystems unstable.

The main contributions are summarized as follows:

1) For the first time, the reachable set estimation problem of switched systems with all subsystems unstable is addressed based on Lyapunov methods. This result naturally applies to the reachable set synthesis of switched systems with unstabilizable subsystems.
2) To avoid the infeasible case that constant Lyapunov matrices would lead to, time-varying Lyapunov matrix functions are introduced to derive the feasible reachable set estimation conditions.
3) Linear interpolations dependent on upper and lower bounds of dwell time are applied to construct the time-varying matrix functions. Not only does this benefit the analysis of switching instants, but it also achieves conservatism reduction.

Notations: Euclidean space with n-dimension is denoted by $\mathbb{R}^n$. The superscript "T" stands for the matrix transposition. Matrix $P > 0$ ($P \geq 0$) means that P is real symmetric and positive definite (semi-definite). The symbol $\mathcal{D}^+$ represents the upper right Dini derivative. The notation "*" denotes an ellipsis of terms that are induced by symmetry. For matrix M, $\textbf{sym}(M) \triangleq M^T + M$.

II. Preliminaries and Problem Formulation

Consider a switched linear system with disturbance input:

$$\dot{x}(t) \;=\; A_{\sigma(t)}x(t) + B_{\omega,\sigma(t)}\omega(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $\omega(t) \in \mathbb{R}^{n_\omega}$ is the disturbance input vector. The disturbance input is assumed to satisfy

$$\omega^T(t)\omega(t) \leq \bar{\omega}^2, \quad \forall t \geq 0, \tag{2}$$

where $\bar{\omega} > 0$ is a known scalar. The symbol $\sigma(t)$ represents the switching law, which is a piecewise constant function taking values in a finite set $\mathcal{N} \triangleq \{1, 2, \ldots, S\}$. A_i and B_i, $i \in \mathcal{N}$, are constant real matrices. The considered case is that all subsystems are unstable, which means that all the subsystem matrices A_i, $i \in \mathcal{N}$ are non-Hurwitz. The switching sequence is denoted as $\{t_0, t_1, \ldots, t_k, \ldots\}$, where t_0 is the initial time. Then each dwell time interval of the subsystem is denoted as $[t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$, and the dwell time is obtained as $\tau_k = t_{k+1} - t_k$.

Concerning the original system, the reachable set is given as

$$\mathcal{R}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x(t_0) = 0, x(t), \omega(t) \text{ satisfy}(1),(2), t \geq 0\}. \tag{3}$$

The main objective of the reachable set estimation is to find a region as small as possible to bound the reachable set $\mathcal{R}_x$. To estimate the reachable set of system (1), the following bounding ellipsoid is introduced:

$$E(P) \triangleq \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq 1, P > 0\}. \tag{4}$$

Just as [12], the dwell time should meet the condition $\tau_k \in [\tau_{\min}, \tau_{\max}]$ with $\tau_{\max} \geq \tau_{\min} > 0$ and $\tau_{\min} \triangleq \inf_{k=0,1,2,\ldots} \tau_k$, $\tau_{\max} \triangleq \sup_{k=0,1,2,\ldots} \tau_k$. Such dwell time constraint is essential as $\tau_{\min} > 0$ prevents zero dwell time that will lead to instability for a switched system consisting of two unstable subsystems, while the upper bound $\tau_{\max}$ prevents excessively large, even infinite dwell time of a subsystem that will make it impossible to be stabilized. Therefore, $\tau_k \in [\tau_{\min}, \tau_{\max}]$ can avoid some scenarios that are unable to stabilize and facilitate developing stability conditions for switched systems with all subsystems unstable. Moreover, as the considered switched systems exhibit linear dynamics, the system instability will result in the unboundedness of the reachable set. Therefore, the dwell time τ_k within a bounded interval $[\tau_{\min}, \tau_{\max}]$ is considered in this work.

By allowing a bounded increment of Lyapunov functions at the switching instants, inspired by [34], Lemma 1 is developed to provide the basic tool for estimating the reachable set of switched systems with all subsystem dwell time within $[\tau_{\min}, \tau_{\max}]$.

Lemma 1. *Consider switched system (1) with the bounded disturbance input (2). Let $V_i(t, x(t))$, $i \in \mathcal{N}$, be a set of continuous functions, satisfying $\forall t \geq 0$, $V_i(t, x(t)) > 0$ for*

$x \in \mathbb{R}^{n_x} \setminus \{0\}$, and $V_i(t,0) = 0$ for only $x(t) = 0$. If there exist scalars $\alpha > 0$, $\mu \geq 1$ such that

$$\mathcal{D}^+V_i(t,x(t)) + \alpha V_i(t,x(t)) - \frac{\alpha}{\bar{\omega}^2}\omega^T(t)\omega(t) \leq 0, \tag{5}$$

$$V_j(t_k,x(t_k)) \leq \mu V_i(t_k^-,x(t_k^-)), i \neq j, \forall i,j \in \mathcal{N}, \tag{6}$$

$$\alpha\tau_{\min} - \ln\mu > 0, \tag{7}$$

where t_k^- denotes the last moment of previous subsystems at the switching instant and $x(t_k) = x(t_k^-)$ due to the continuous state. Then under zero initial conditions, one has $V_i(t,x(t)) \leq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min}-\ln\mu}$, $\forall i \in \mathcal{N}$.

Proof. Consider any $t \in [t_k, t_{k+1})$, $V(t,x(t)) = V_i(t,x(t))$, $i \in \mathcal{N}$. With $\omega^T(t)\omega(t) \leq \bar{\omega}^2$, condition (5) gives that

$$\mathcal{D}^+V(t,x(t)) + \alpha V(t,x(t)) \leq \alpha, \quad t \in [t_k,t_{k+1}). \tag{8}$$

Multiplying (8) with $e^{\alpha(t-t_k)}$, it follows that

$$e^{\alpha(t-t_k)}(\mathcal{D}^+V(t,x(t)) + \alpha V(t,x(t))) \leq \alpha e^{\alpha(t-t_k)}. \tag{9}$$

Then, integrating (9) from t_k to t and using the fundamental theorem of calculus, one can get that

$$\begin{aligned} & V(t,x(t)) \\ & \leq e^{-\alpha(t-t_k)}V(t_k,x(t_k)) + \alpha\int_{t_k}^{t} e^{-\alpha(t-\tau)}d\tau \\ & \leq \mu^k e^{-\alpha(t-t_0)}V(t_0,x(t_0)) + \alpha\sum_{j=1}^{k}\mu^j e^{-\alpha(t-t_{k+1-j})} \\ & \quad \times \int_{t_{k-j}}^{t_{k+1-j}} e^{-\alpha(t_{k+1-j}-\tau)}d\tau + \alpha\int_{t_k}^{t} e^{-\alpha(t-\tau)}d\tau. \end{aligned} \tag{10}$$

Under zero initial conditions, $x(t_0) = 0$ gives that $V(t_0,x(t_0)) = V(t_0,0) = 0$. Then it can be obtained that

$$\begin{aligned} V(t,x(t)) \leq & \alpha\sum_{j=1}^{k}\int_{t_{k-j}}^{t_{k+1-j}} \mu^j e^{-\alpha(t-\tau)}d\tau \\ & + \alpha\int_{t_k}^{t} e^{-\alpha(t-\tau)}d\tau. \end{aligned} \tag{11}$$

It should be noted that the index j of μ^j counts the switching numbers in the time interval $[\tau, t)$, where $t \in [t_k, t_{k+1})$, $\tau \in [t_{k-j}, t_{k+1-j})$, $j = 1, 2, \ldots, k$, $k = 1, 2, \ldots$. Referring to [35], with the lower bound of dwell time $\tau_{\min}$, it follows $j \leq (1 + \frac{t-\tau}{\tau_{\min}})$. Moreover, with $\tau \in [t_k, t)$, it holds that $1 \leq \mu^{1+\frac{t-\tau}{\tau_{\min}}}$. Then, it can be obtained that

$$\begin{aligned} V(t,x(t)) & \leq \alpha\sum_{j=1}^{k}\int_{t_{k-j}}^{t_{k+1-j}} \mu^{1+\frac{t-\tau}{\tau_{\min}}} e^{-\alpha(t-\tau)}d\tau \\ & \quad + \alpha\int_{t_k}^{t} \mu^{1+\frac{t-\tau}{\tau_{\min}}} e^{-\alpha(t-\tau)}d\tau \\ & = \alpha\int_{t_0}^{t} \mu^{1+\frac{t-\tau}{\tau_{\min}}} e^{-\alpha(t-\tau)}d\tau \\ & = \frac{\alpha\mu}{\alpha - \frac{\ln\mu}{\tau_{\min}}}(1 - e^{-(\alpha-\frac{\ln\mu}{\tau_{\min}})t}). \end{aligned} \tag{12}$$

With $\alpha\tau_{\min} - \ln\mu > 0$, one can get that $(1 - e^{-(\alpha-\frac{\ln\mu}{\tau_{\min}})t}) \leq 1$, thus

$$V(t,x(t)) \leq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}. \tag{13}$$

Therefore, (5)–(7) can ensure that $V(t,x(t)) \leq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min}-\ln\mu}$, which further gives $V_i(t,x(t)) \leq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min}-\ln\mu}$, $i \in \mathcal{N}$. □

Remark 1. *The above lemma does not restrict any nature or form of the subsystem dynamics, as the proof does not involve the specific structure of the subsystem. It is noted that the initial time t_0 does not introduce μ in (6) since t_0^- is beyond the domain of our interest. The derived upper bound of Lyapunov function $V_i(t,x(t))$, $i \in \mathcal{N}$, is dependent on the lower bound of dwell time. It is reasonable since the jumps at the switching instants should be accumulated with the upper limit of the switching numbers. The dwell time setting of this work is different from the average dwell time in [34]. Moreover, the considered Lyapunov function $V_i(t,x(t))$ will be constructed by system state and time-varying symmetric positive definite matrices in the subsequent derivations. Thus, $\mathcal{D}^+V_i(t,x(t))$ will introduce the derivatives related to time-varying matrix functions, which is different from [34].*

III. Reachable Set Estimation

In this section, we will construct novel time-varying Lyapunov matrix functions to develop implementable reachable set estimation conditions for switched systems with all subsystems unstable.

We adopted a Lyapunov candidate with time-varying Lyapunov matrix function $P_i(t)$ as follows:

$$V_i(t,x(t)) = x^T(t)P_i(t)x(t), \quad t \in [t_k,t_{k+1}), \tag{14}$$

where $P_i(t) > 0$ is continuous, and $i \in \mathcal{N}$, $k = 0, 1, 2, \ldots$. Then the reachable set estimation based on Lemma 1 is developed in the following theorem.

Theorem 1. *Consider switched system with all subsystems unstable (1) under the disturbance input (2). If there exist a set of continuous matrix functions $P_i(t) > 0$, $i \in \mathcal{N}$, matrix $\hat{P} > 0$, and scalars $\alpha > 0$, $\mu \geq 1$, such that (7) and*

$$\begin{bmatrix} \Gamma_i(t) & P_i(t)B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2}I \end{bmatrix} < 0, \tag{15}$$

$$P_j(t_k) \leq \mu P_i(t_k^-), \; i \neq j, \; \forall i,j \in \mathcal{N}, \tag{16}$$

$$P_i(t) \geq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}\hat{P}, \tag{17}$$

hold, where $\Gamma_i(t) = \mathcal{D}^+P_i(t) + A_i^TP_i(t) + P_i(t)A_i + \alpha P_i(t)$, then the reachable set $\mathcal{R}_x$ of system (1) can be enclosed by the ellipsoid $E(\hat{P})$.

Proof. With the Lyapunov candidate defined in (14) and along the trajectory of system (1), one has

$$\begin{aligned} & \mathcal{D}^+V_i(t,x(t)) + \alpha V_i(t,x(t)) - \frac{\alpha}{\bar{\omega}^2}\omega^T(t)\omega(t) \\ & = \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}^T \begin{bmatrix} \Gamma_i(t) & P_i(t)B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2}I \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}. \end{aligned} \tag{18}$$

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where $\Gamma_i(t) = \mathcal{D}^+ P_i(t) + A_i^T P_i(t) + P_i(t)A_i + \alpha P_i(t)$. Then from condition (15), it can be obtained that

$$\mathcal{D}^+ V_i(t, x(t)) + \alpha V_i(t, x(t)) - \frac{\alpha}{\bar{\omega}^2}\omega^T(t)\omega(t) \leq 0. \quad (19)$$

At the switching instants t_k, with continuous $x(t)$, condition (16) generates that $V_j(t_k, x(t_k)) \leq \mu V_i(t_k^-, x(t_k^-)), i \neq j, \forall i, j \in \mathcal{N}$.

Therefore, conditions (15)–(16) can guarantee (5)–(6) in Lemma 1 hold. According to Lemma 1, one can obtain that $V_i(t, x(t)) \leq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}$, $i \in \mathcal{N}$, that is,

$$V_i(t, x(t)) = x^T(t)P_i(t)x(t) \leq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}. \quad (20)$$

If there exists a matrix $\hat{P} > 0$ satisfying

$$P_i(t) \geq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}\hat{P}, \quad (21)$$

and with (20), it follows that

$$\frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu} \geq x^T(t)P_i(t)x(t) \geq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}x^T(t)\hat{P}x(t). \quad (22)$$

Thus one can get that

$$x^T(t)\hat{P}x(t) \leq 1, \quad (23)$$

and the reachable set $\mathcal{R}_x$ of system (1) can be enclosed by the ellipsoid $E(\hat{P}) = \{x \in \mathbb{R}^{n_x} \mid x^T\hat{P}x \leq 1, \hat{P} > 0\}$. □

Given the scenario of all subsystems being unstable, the reason for using time-varying Lyapunov matrices rather than commonly used constant Lyapunov matrices is discussed in the following remark.

Remark 2. *Regarding each time instant t, in terms of Schur Complement equivalence, $\Gamma_i(t) < 0$ is required to ensure that (15) holds, that is,*

$$\mathcal{D}^+ P_i(t) + A_i^T P_i(t) + P_i(t)A_i + \alpha P_i(t) < 0. \quad (24)$$

Inequality (24) can be rewritten as follows:

$$\mathcal{D}^+ P_i(t) + (A_i + \frac{1}{2}\alpha I)^T P_i(t) + P_i(t)(A_i + \frac{1}{2}\alpha I) < 0. \quad (25)$$

By referring [36], inequity (25) admits existence of $P_i(t)$ for non-Hurwitz A_i, $i \in \mathcal{N}$. However, it is noted that the frequently employed Lyapunov candidates for switched systems are multiple Lyapunov functions, that is $V_i(x(t)) = x^T(t)P_ix(t)$. With this type of Lyapunov function, (24) becomes

$$A_i^T P_i + P_iA_i + \alpha P_i < 0. \quad (26)$$

With A_i being non-Hurwitz (unstable subsystem) and $\alpha > 0$, (26) is infeasible to find matrices $P_i > 0$. The discretized Lyapunov function constructed in [12] is also not applicable in our reachable set estimation problem. Specifically, $V_i(t) = x^T(t)P_{i,L}x(t)$, $t \in [t_k + \tau_{\min}, t_{k+1})$ in [12] will also be infeasible with $P_{i,L} > 0$.

Although Theorem 1 develops a reachable set estimation condition for system (1), it is difficult to numerically check the existence of matrix function $P_i(t)$, $i \in \mathcal{N}$. Therefore, we will construct novel specific forms of time-varying matrix functions by exploiting the upper and lower bound information of the dwell time to transform the untractable time-varying linear matrix inequalities (LMIs) into implementable LMI conditions. Notably, Theorem 1 serves as an intermediate result to demonstrate the advantages of using time-varying matrix function $P_i(t)$ and generates the estimated ellipsoid $E(\hat{P})$ enclosing the reachable set $\mathcal{R}_x$. The subsequent results present tractable reachable set estimation conditions, enabling effective computation and optimization.

Since the dwell time of each unstable subsystem belongs to the interval $[\tau_{\min}, \tau_{\max}]$, and $\tau_{\max} \geq \tau_{\min}$, two scenarios are considered as follows.

Case 1 ($\tau_{\max} > \tau_{\min}$): We will construct a continuous time-varying Lyapunov matrix function for i^{th} subsystem using the dwell time boundary information as follows:

$$P_i(t) = \begin{cases} P_{i,1}(t), & t \in [t_k, t_k + \tau_{\min}), \\ P_{i,2}(t), & t \in [t_k + \tau_{\min}, t_k + \tau_{\max}), \end{cases} \quad i \in \mathcal{N}. \quad (27)$$

The continuity of $P_i(t)$ is ensured through $P_{i,1}(t)$ and $P_{i,2}(t)$, with $P_{i,1}(t)$ and $P_{i,2}(t)$ being continuous and $\lim_{t \to t_k + \tau_{\min}} P_{i,1}(t) = P_{i,2}(t_k + \tau_{\min})$.

It is intuitive and effective to use linear interpolation methods to construct the specific form of $P_{i,1}(t)$ and $P_{i,2}(t)$. With dividing the interval $[t_k, t_k + \tau_{\min})$ into M segments and denoting $\Delta_1 = \frac{\tau_{\min}}{M}$, the form of $P_{i,1}(t)$ is given as

$$P_{i,1}(t) = P_{i,1,m} + (t - t_k - m\Delta_1)\frac{P_{i,1,m+1} - P_{i,1,m}}{\Delta_1},$$
$$t \in [t_k + m\Delta_1, t_k + (m+1)\Delta_1), \quad (28)$$

and then, dividing the interval $[t_k + \tau_{\min}, t_k + \tau_{\max})$ into M segments and denoting $\Delta_2 = \frac{\tau_{\max} - \tau_{\min}}{M}$, the form of $P_{i,2}(t)$ is given as

$$P_{i,2}(t) = P_{i,2,m} + (t - t_k - \tau_{\min} - m\Delta_2)\frac{P_{i,2,m+1} - P_{i,2,m}}{\Delta_2},$$
$$t \in [t_k + \tau_{\min} + m\Delta_2, t_k + \tau_{\min} + (m+1)\Delta_2), \quad (29)$$

To guarantee the continuity of $P_i(t)$, $P_{i,1,M} = P_{i,2,0}$ holds.

A demonstration for illustrating the evolution of the constructed Lyapunov matrix function $P_i(t)$ is displayed in the following sketch.

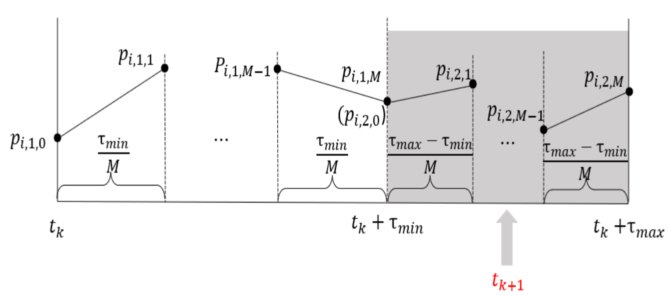

Fig. 1: Sketch of $P_i(t)$ when $\tau_{\max} > \tau_{\min}$

With the constructed time-varying matrix functions in (27)–(29), the following theorem develops sufficient conditions based on Theorem 1 for tractable reachable set estimation.

Theorem 2. *Consider switched system with all subsystems unstable (1). If there exist matrices* $P_{i,1,m} > 0$, $P_{i,2,m} > 0$, $i \in \mathcal{N}$, $m = 0, 1, \ldots, M$, *with* $P_{i,1,M} = P_{i,2,0}$, $\hat{P} > 0$, *and scalars* $\alpha > 0$, $\mu \geq 1$, *such that (7) and*

$$\begin{bmatrix} \frac{P_{i,1,m+1}-P_{i,1,m}}{\Delta_1} + \Omega(A_i, P_{i,1,m}, \alpha) & P_{i,1,m}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} < 0, \tag{30}$$

$$\begin{bmatrix} \frac{P_{i,1,m+1}-P_{i,1,m}}{\Delta_1} + \Omega(A_i, P_{i,1,m+1}, \alpha) & P_{i,1,m+1}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} \frac{P_{i,2,m+1}-P_{i,2,m}}{\Delta_2} + \Omega(A_i, P_{i,2,m}, \alpha) & P_{i,2,m}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} \frac{P_{i,2,m+1}-P_{i,2,m}}{\Delta_2} + \Omega(A_i, P_{i,2,m+1}, \alpha) & P_{i,2,m+1}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} < 0, \tag{33}$$

$$P_{j,1,0} \leq \mu P_{i,2,m}, \; i \neq j, \; \forall i, j \in \mathcal{N}, \tag{34}$$

$$P_{i,1,m} \geq \frac{\mu \alpha \tau_{\min}}{\alpha \tau_{\min} - \ln \mu} \hat{P}, \tag{35}$$

$$P_{i,2,m} \geq \frac{\mu \alpha \tau_{\min}}{\alpha \tau_{\min} - \ln \mu} \hat{P}, \tag{36}$$

hold, where $\Omega(A, P, \alpha) \triangleq \mathbf{sym}(PA) + \alpha P$, *then the reachable set* $\mathcal{R}_x$ *of system (1) can be enclosed by the ellipsoid* $E(\hat{P})$.

Proof. Consider any $t \in [t_k, t_{k+1})$, subsystem i, $i \in \mathcal{N}$ is activated. For i^{th} subsystem, $V_i(t, x(t)) = x^T(t)P_i(t)x(t)$ with $P_i(t)$ in (27).

When $t \in [t_k, t_k + \tau_{\min})$, it follows that $V_i(t, x(t)) = x^T(t)P_{i,1}(t)x(t)$, where $P_{i,1}(t)$ is given by (28), then

$$\begin{aligned} &\begin{bmatrix} \mathcal{D}^+ P_{i,1}(t) + \mathbf{sym}\big(P_{i,1}(t)A_i\big) + \alpha P_{i,1}(t) & P_{i,1}(t)B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} \\ &= \left(1 - \frac{t - t_k - m\Delta_1}{\Delta_1}\right) \begin{bmatrix} \mathcal{M}_{i,1,m} & P_{i,1,m}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} \\ &\quad + \left(\frac{t - t_k - m\Delta_1}{\Delta_1}\right) \begin{bmatrix} \mathcal{M}_{i,1,m+1} & P_{i,1,m+1}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix}, \end{aligned} \tag{37}$$

where

$$\mathcal{M}_{i,j,m} \triangleq \frac{P_{i,j,m+1} - P_{i,j,m}}{\Delta_j} + \Omega(A_i, P_{i,j,m}, \alpha),$$

$$\mathcal{M}_{i,j,m+1} \triangleq \frac{P_{i,j,m+1} - P_{i,j,m}}{\Delta_j} + \Omega(A_i, P_{i,j,m+1}, \alpha).$$

Then, since $0 \leq \frac{t - t_k - m\Delta_1}{\Delta_1} < 1$, $t \in [t_k + m\Delta_1, t_k + (m+1)\Delta_1)$, inequalities (30) and (31) can guarantee that

$$\begin{bmatrix} \mathcal{D}^+ P_{i,1}(t) + \mathbf{sym}\big(P_{i,1}(t)A_i\big) + \alpha P_{i,1}(t) & P_{i,1}(t)B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} < 0. \tag{38}$$

When $t \in [t_k + \tau_{\min}, t_{k+1})$, it holds that $t_{k+1} \leq t_k + \tau_{\max}$. Consider $V_i(t, x(t)) = x^T(t)P_{i,2}(t)x(t)$, $t \in [t_k + \tau_{\min}, t_k + \tau_{\max})$, one has

$$\begin{aligned} &\begin{bmatrix} \mathcal{D}^+ P_{i,2}(t) + \mathbf{sym}\big(P_{i,2}(t)A_i\big) + \alpha P_{i,2}(t) & P_{i,2}(t)B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} \\ &= \left(1 - \frac{t - t_k - \tau_{\min} - m\Delta_2}{\Delta_2}\right) \begin{bmatrix} \mathcal{M}_{i,2,m} & P_{i,1,m}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} + \\ &\quad \left(\frac{t - t_k - \tau_{\min} - m\Delta_2}{\Delta_2}\right) \begin{bmatrix} \mathcal{M}_{i,2,m+1} & P_{i,1,m+1}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix}. \end{aligned} \tag{39}$$

Then with $0 \leq \frac{t - t_k - \tau_{\min} - m\Delta_2}{\Delta_2} < 1$ for $t \in [t_k + \tau_{\min} + m\Delta_2, t_k + \tau_{\min} + (m+1)\Delta_2)$, inequalities (32) and (33) can guarantee that

$$\begin{bmatrix} \mathcal{D}^+ P_{i,2}(t) + \mathbf{sym}\big(P_{i,2}(t)A_i\big) + \alpha P_{i,2}(t) & P_{i,2}(t)B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2} I \end{bmatrix} < 0. \tag{40}$$

Since $t_{k+1} \leq t_k + \tau_{\max}$, the conditions that hold over the interval $[t_k, t_k + \tau_{\max})$ are naturally satisfied over the subinterval $[t_k, t_{k+1})$. Therefore, (38) and (40) ensure that (15) holds for $t \in [t_k, t_{k+1})$ with $P_i(t)$ given by (27). In summary, (30)–(33) guarantee that condition (15) in Theorem 1 holds.

At the switching instants, $P_j(t_k) \leq \mu P_i(t_k^-)$ in (16) should hold. However, the switching instants t_{k+1}, $k = 0, 1, 2, \ldots$, cannot be exactly known, while located in $[t_k + \tau_{\min}, t_k + \tau_{\max})$, as shown in Fig. 1. It is known that the initial Lyapunov matrix of the next activated subsystem j is $P_{j,1,0}$, while the last instant for switching of previous subsystem i located in $[t_k + \tau_{\min}, t_k + \tau_{\max})$ with $P_{i,2}(t)$ defined in linear interpolation form (29), thus *although the Lyapunov matrix at the switching instant cannot be determined, it must be one linear combination of adjacent vertices* ($P_{i,2,m}$, $m = 0, 1, \ldots, M$) *of* $P_{i,2}(t)$. Thus, condition (34) can guarantee that (16) holds.

Additionally, with $P_i(t)$ defined by the linear interpolation in (27)–(29), inequalities (35) and (36) ensure that (17) holds.

Therefore, according to Theorem 1, one can get that $x^T(t)\hat{P}x(t) \leq 1$. It is concluded that the reachable set can be enclosed by the ellipsoid $E(\hat{P})$. □

Case 2 ($\tau_{\max} = \tau_{\min}$): The dwell time of each unstable subsystem is fixed. The Lyapunov matrix function for i^{th} subsystem will reduce to

$$\begin{aligned} P_i(t) &= P_{i,1}(t) \\ &= P_{i,1,m} + (t - t_k - m\Delta_1)\frac{P_{i,1,m+1} - P_{i,1,m}}{\Delta_1}, \\ & t \in [t_k + m\Delta_1, t_k + (m+1)\Delta_1). \end{aligned} \tag{41}$$

The evolution of $P_i(t)$ is demonstrated in Fig. 2.

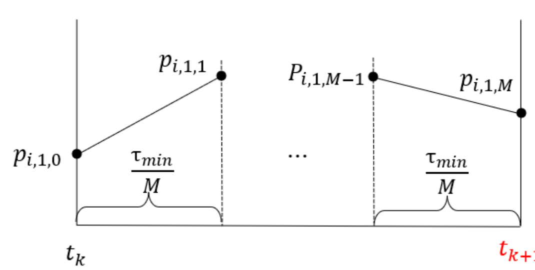

Fig. 2: Sketch of $P_i(t)$ when $\tau_{\max} = \tau_{\min}$

In this case, the Lyapunov matrix function is the same as those in [12], and the reachable set estimation condition is established in Corollary 1.

Corollary 1. *Consider switched system with all subsystems unstable (1). If there exist matrices $P_{i,1,m} > 0$, $\hat{P} > 0$, and scalars $\alpha > 0$, $\mu \geq 1$, such that (7) and*

$$\begin{bmatrix} \frac{P_{i,1,m+1}-P_{i,1,m}}{\Delta_1} + \Omega(A_i, P_{i,1,m}, \alpha) & P_{i,1,m}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2}I \end{bmatrix} < 0, \tag{42}$$

$$\begin{bmatrix} \frac{P_{i,1,m+1}-P_{i,1,m}}{\Delta_1} + \Omega(A_i, P_{i,1,m+1}, \alpha) & P_{i,1,m+1}B_{\omega,i} \\ * & -\frac{\alpha}{\bar{\omega}^2}I \end{bmatrix} < 0, \tag{43}$$

$$P_{j,1,0} \leq \mu P_{i,1,M}, \ i \neq j, \ \forall i,j \in \mathcal{N}, \tag{44}$$

$$P_{i,1,m} \geq \frac{\mu\alpha\tau_{\min}}{\alpha\tau_{\min} - \ln\mu}\hat{P}, \tag{45}$$

hold, then the reachable set $\mathcal{R}_x$ can be enclosed by the ellipsoid $E(\hat{P})$.

Proof. Since $\tau_{\max} = \tau_{\min}$, for any $t \in [t_k, t_{k+1})$, subsystem i, $i \in \mathcal{N}$, is activated. With the constructed Lyapunov function $V_i(t, x(t)) = x^T(t)P_i(t)x(t)$ in (41) for i^{th} subsystem. Since $t_{k+1} = t_k + \tau_{\min}$, when $t \in [t_k, t_{k+1})$, it can be obtained that $V_i(t, x(t)) = x^T(t)P_i(t)x(t) = x^T(t)P_{i,1}(t)x(t)$. By following the similar proof of Theorem 2, one can get that (42)–(43) can ensure condition (15) holds. At the switching instants, it is obtained that $P_{j,1,0} \leq \mu P_{i,1,M}$, $i \neq j$, $\forall i,j \in \mathcal{N}$ can ensure condition (16) holds. Additionally, condition (45) is derived from condition (17). In summary, based on Theorem 1, conditions (42)–(45) ensure that the reachable set $\mathcal{R}_x$ of system (1) is enclosed by the ellipsoid $E(\hat{P})$. □

Remark 3. *Our approach for reachable set estimation is specifically designed to handle cases where all subsystems are unstable. The focus is to address the limitation of existing Lyapunov methods when confronted with the infeasible issue of estimating the reachable set for unstable subsystems. However, it can also be effectively applied to switched systems that include both stable and unstable subsystems. If the dwell time setting of all stable and unstable subsystems is still within $[\tau_{\min}, \tau_{\max}]$, the derived results can be directly extended to such scenarios.*

IV. OPTIMIZATION

It is noted that the bounding ellipsoid $E(\hat{P})$ with $\hat{P} > 0$ obtained in Theorem 2 and Corollary 1 can enclose the reachable set of switched system (1). In general, the ellipsoid is desired to be as small as possible. Consequently, it needs to develop optimization methods aimed at minimizing its size. Herein, two optimization approaches are developed below.

a) *Minimizing the major axis of the ellipsoid*: Introducing a variable $\varepsilon > 0$, with $\hat{P} > 0$, it holds that

$$\hat{P} \geq \varepsilon I, \tag{46}$$

which implies $1 \geq x^T\hat{P}x \geq \varepsilon x^T x$, thus the ellipsoid $E(\hat{P})$ is contained in a ball $\mathcal{B}(\varepsilon) = \{x \in \mathbb{R}^{n_x} \mid \varepsilon x^T x \leq 1\}$. By referring to [27], the major axis of the ellipsoid $E(\hat{P})$ could be minimized by maximizing ε. By letting $\epsilon = \varepsilon^{-1}$, (46) can be rewritten as

$$\begin{bmatrix} \hat{P} & I \\ * & \epsilon I \end{bmatrix} \geq 0, \quad \epsilon > 0. \tag{47}$$

Then, the optimization problem can be formulated as

$$\begin{aligned} &\text{minimize } \epsilon \\ &\text{subject to} \begin{cases} (7),(30)-(36),(47) \text{ for Theorem 2,} \\ (7),(42)-(45),(47) \text{ for Corollary 1.} \end{cases} \end{aligned} \tag{48}$$

b) *Minimizing the volume of the ellipsoid*: The volume of the ellipsoid $E(\hat{P})$ is proportional to the product of all semi-axial lengths. Additionally, the semi-axial length of $E(\hat{P})$ is the square root of the reciprocal eigenvalue of $\hat{P}$. Thus, the volume of $E(\hat{P})$ is proportional to $-\ln(\det(\hat{P}))$. To find an ellipsoid $E(\hat{P})$ with the volume as small as possible, the optimization problem is formulated as follows:

$$\begin{aligned} &\text{minimize } -\ln(\det(\hat{P})) \\ &\text{subject to} \begin{cases} (7),(30)-(36) \text{ for Theorem 2,} \\ (7),(42)-(45) \text{ for Corollary 1.} \end{cases} \end{aligned} \tag{49}$$

Remark 4. *Two optimization methods have been developed to minimize the size of the estimated ellipsoid. Minimizing the major axis of the ellipsoid is commonly used in most studies, while minimizing the volume is an alternative method to reduce the region size [19]. These optimized objectives can be tailored to meet specific requirements or accommodate physical constraints. With the given $\tau_{\min}$ and fixed α, μ in Theorem 2 and Corollary 1, the developed optimization problems become convex as the conditions are expressed in terms of LMIs. The number of decision variables of Theorem 2 and Corollary 1 are $(2MS + S + 1)n_x(n_x + 1)/2$ and $(MS + S + 1)n_x(n_x + 1)/2$, respectively, which is related to the computational complexity. With increasing values of M and S, more decision variables will be introduced to increase the computational burden.*

V. ILLUSTRATIVE EXAMPLE

Consider a switched system provided in [12] consisting of two unstable subsystems with the following parameters:

$$A_1 = \begin{bmatrix} -1.9 & 0.6 \\ 0.6 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & -0.9 \\ 0.1 & -1.4 \end{bmatrix},$$

and in system (1) with disturbance input, the parameters are given as $B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $B_2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$. Disturbance signals $\omega(t) = 1$, $\omega(t) = -1$ and $\omega(t) = \sin t$ with bounded peak $\bar{\omega} = 1$ are considered.

Two cases of dwell time confined in $[\tau_{\min}, \tau_{\max}]$ with $\tau_{\max} \geq \tau_{\min} > 0$ are conducted for simulation illustration. The simulation is achieved by MATLAB R2019b, using YALMIP with solver MOSEK.

Case 1 ($\tau_{\max} > \tau_{\min}$): When $\tau_{\max} > \tau_{\min}$, it is given that $\tau_{\max} = 4$, $\tau_{\min} = 2$. Then with $\alpha = 0.2$, $\mu = 1$, performing two optimization objectives for Theorem 2, the results with increasing values of M are obtained in TABLEs I and II. It can be seen that as the value of M increases, the values of the two optimal objectives ϵ and $-\ln(\det(\hat{P}))$ become smaller and smaller, while the computation time is getting longer. This indicates that more segmentation of linear interpolations for constructing time-varying Lyapunov matrix functions can reduce conservatism in finding the bounding regions of the reachable set. Moreover, the discretized Lyapunov method in [12] results in infeasible solutions of reachable set estimation, which is consistent with the discussion in Remark 2. Furthermore, a switching signal with satisfying $\tau_{\max} = 4$, $\tau_{\min} = 2$ is depicted in Fig. 3. Under this switching signal, the reachable sets of the switched systems with $\omega(t) = 1$, $\omega(t) = -1$ and $\omega(t) = \sin t$ and the bounding regions obtained in TABLE I and TABLE II are demonstrated in Fig. 4 and Fig. 5, respectively. It is obvious that a larger M can give a smaller bounding region. Moreover, optimizing the volume by minimizing $-\ln(\det(\hat{P}))$ could find smaller ellipsoids than only optimizing the major axis by minimizing ϵ.

TABLE I: Optimized ϵ and computation time (C.T.) with different M of Theorem 2

M	1	2	5	10	50	100
ϵ	infeasible	236.9277	30.6998	18.7412	13.3902	12.9034
C.T.	–	1.022 s	1.571 s	2.445 s	22.061 s	42.914 s
ϵ by [12]	infeasible	infeasible	infeasible	infeasible	infeasible	infeasible

TABLE II: Optimized $-\ln(\det(\hat{P}))$ and C.T. with different M of Theorem 2

M	1	5	10	50	100
$-\ln(\det(\hat{P}))$	38.9849	6.7228	5.6909	4.7814	4.6688
C.T.	1.314 s	1.574 s	2.904 s	14.248 s	35.665 s
$-\ln(\det(\hat{P}))$ by [12]	infeasible	infeasible	infeasible	infeasible	infeasible

Case 2 ($\tau_{\max} = \tau_{\min}$): When $\tau_{\min} = \tau_{\max} = 2$, the switching sequence is periodic. Based on Corollary 1 with $\alpha = 0.4$, $\mu = 1.1$, the results are obtained by minimizing ϵ and $-\ln(\det(\hat{P}))$ with different values of M, which is listed in TABLEs III and IV. It can be seen from TABLE III and TABLE IV that with the increase of M value, the optimal objective value decreases, and the computation time increases. In this case, the periodic switching signal is shown in Fig. 6. The reachable sets corresponding to $\omega(t) = 1$, $\omega(t) = -1$ and $\omega(t) = \sin t$ and the obtained bounding ellipsoids $E(\hat{P})$ from TABLE III and TABLE IV are depicted in Fig. 7 and Fig. 8, respectively.

TABLE III: Optimized ϵ and C.T. with different M of Corollary 1

M	1	2	5	10	50	100
ϵ	infeasible	20.3070	6.4914	4.8522	3.9269	3.8263
C.T.	–	0.820 s	0.986 s	2.175 s	5.899 s	13.516 s

TABLE IV: Optimized $-\ln(\det(\hat{P}))$ and C.T. with different M of Corollary 1

M	1	5	10	50	100
$-\ln(\det(\hat{P}))$	40.2940	3.4862	2.7974	2.2575	2.1927
C.T.	0.807 s	1.091 s	1.816 s	7.768 s	19.841 s

VI. Conclusion

This article first investigates the reachable set estimation problem using the Lyapunov approach for a class of switched systems with all subsystems unstable. Addressing the limitations of existing Lyapunov methods in estimating reachable set estimation for unstable subsystems, a novel Lyapunov approach with time-varying Lyapunov matrix functions is proposed to establish feasible reachable set estimation conditions for unstable subsystems. Linear interpolations facilitate the establishment of implementable sufficient conditions in terms of linear matrix inequalities (LMIs). The effectiveness of estimating the reachable set and conservatism reduction of the developed method is verified through numerical simulations. Future work could explore reachable set control under asynchronous switching. It may also investigate reachable set issues for switched multi-agent systems with unstable modes.

References

[1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[2] D. Liberzon, *Switching in Systems and Control*. Boston: Birkhauser: Springer, 2003, vol. 190.

[3] M. M. Rayguru, S. Roy, L. Yi, M. R. Elara, and S. Baldi, "Introducing switched adaptive control for self-reconfigurable mobile cleaning robots," *IEEE Transactions on Automation Science and Engineering*, vol. 21, no. 2, pp. 2051–2062, 2023.

[4] G. Wen, X. Yu, W. Yu, and J. Lu, "Coordination and control of complex network systems with switching topologies: A survey," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 10, pp. 6342–6357, 2020.

[5] C. Fan, J. Lam, K.-F. Chu, X. Lu, and K.-W. Kwok, "Output reachable set-based leader-following consensus of positive agents over switching networks," *IEEE Transactions on Cybernetics*, vol. 54, no. 7, pp. 3918–3930, 2023.

[6] H. Yue, K. Xing, H. Hu, W. Wu, and H. Su, "Resource failure and buffer space allocation control for automated manufacturing systems," *Information Sciences*, vol. 450, pp. 392–408, 2018.

[7] K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2469–2471, 1994.

[8] J. Zhao and D. J. Hill, "A notion of passivity for switched systems with state-dependent switching," *Journal of Control Theory and Applications*, vol. 4, no. 1, pp. 70–75, 2006.

[9] C. Briat, "Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 24, pp. 198–226, 2017.

[10] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No. 99CH36304)*, vol. 3. IEEE, 1999, pp. 2655–2660.

[11] S. Shi, Z. Shi, and Z. Fei, "Asynchronous control for switched systems by using persistent dwell time modeling," *Systems & Control Letters*, vol. 133, p. 104523, 2019.

[12] W. Xiang and J. Xiao, "Stabilization of switched continuous-time systems with all modes unstable via dwell time switching," *Automatica*, vol. 50, no. 3, pp. 940–945, 2014.

[13] Y.-E. Wang, H. R. Karimi, and D. Wu, "Conditions for the stability of switched systems containing unstable subsystems," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 66, no. 4, pp. 617–621, 2018.

[14] A.-Y. Lu and G.-H. Yang, "Stabilization of switched systems with all modes unstable via periodical switching laws," *Automatica*, vol. 122, p. 109150, 2020.

[15] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[16] J. M. Bravo, T. Alamo, and E. F. Camacho, "Robust MPC of constrained discrete-time nonlinear systems based on approximated reachable sets," *Automatica*, vol. 42, no. 10, pp. 1745–1751, 2006.

[17] W. Lin, Z. Yang, and Z. Ding, "Reachable set estimation and safety verification of nonlinear systems via iterative sums of squares programming," *Journal of Systems Science and Complexity*, vol. 35, no. 3, pp. 1154–1172, 2022.

[18] Y. Zhou and J. S. Baras, "Reachable set approach to collision avoidance for UAVs," in *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, 2015, pp. 5947–5952.

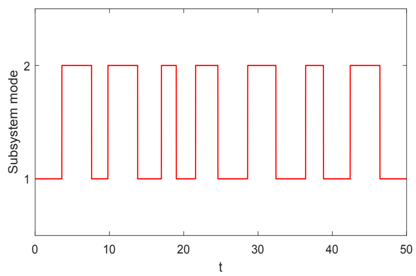

Fig. 3: Switching signal of Case 1

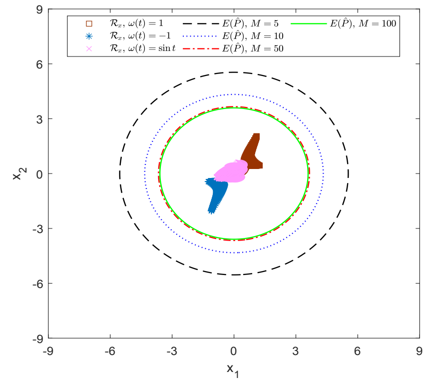

Fig. 4: Reachable sets and bounding ellipsoids by minimizing ϵ of Case 1

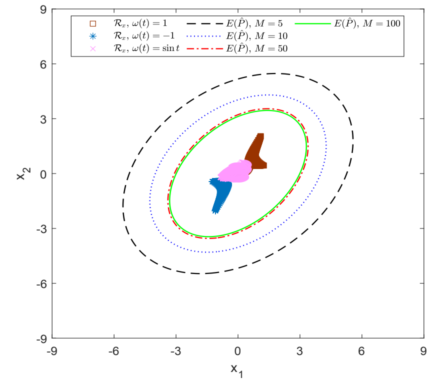

Fig. 5: Reachable sets and bounding ellipsoids by minimizing $-\ln(\det(\hat{P}))$ of Case 1

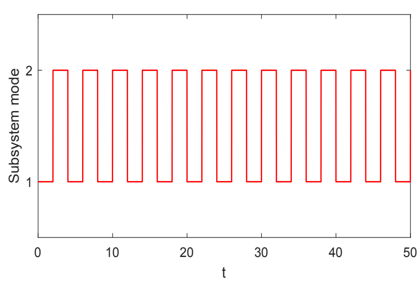

Fig. 6: Switching signal of Case 2

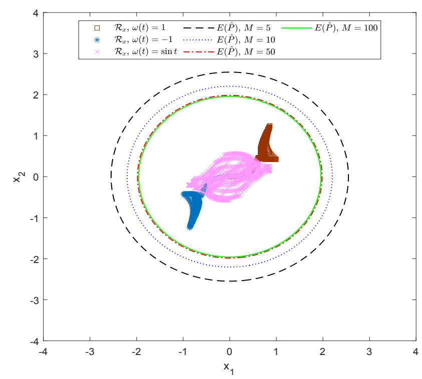

Fig. 7: Reachable sets and bounding ellipsoids by minimizing ϵ of Case 2

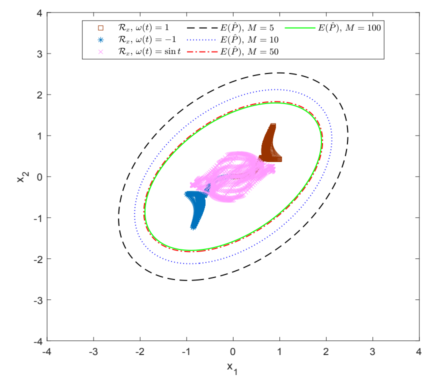

Fig. 8: Reachable sets and bounding ellipsoids by minimizing $-\ln(\det(\hat{P}))$ of Case 2

[19] X. Xie, C. Fan, K.-W. Kwok, and J. Lam, "A Bernstein polynomial approach to estimating reachable set of periodic piecewise polynomial systems," *IEEE Transactions on Automatic Control*, vol. 66, no. 10, pp. 4812–4819, 2020.

[20] X. Jiang, G. Xia, Z. Feng, W. X. Zheng, and R. Su, "Reachable set estimation of multiagent systems: An approximate consensus perspective," *IEEE Transactions on Control of Network Systems*, vol. 11, no. 1, pp. 353–363, 2023.

[21] J. Wang, L. Zhang, X. Zhao, and N. Zhao, "Improved results on reachable set estimation for singularly perturbed systems with time-varying delay," *Journal of the Franklin Institute*, vol. 361, no. 1, pp. 125–134, 2024.

[22] W. Chang, L. Wu, S. Zhu, H. Sang, and L. Guo, "Improved switching condition for reachable set estimation of discrete-time switched delayed neural networks," *Neural Networks*, vol. 179, p. 106530, 2024.

[23] X. Song, J. Lam, B. Zhu, and C. Fan, "Interval observer-based fault-tolerant control for a class of positive Markov jump systems," *Information Sciences*, vol. 590, pp. 142–157, 2022.

[24] F. Zhu, Y. Fu, and T. N. Dinh, "Asymptotic convergence unknown input observer design via interval observer," *Automatica*, vol. 147, p. 110744, 2023.

[25] F. Zhu and M. Li, "Distributed interval observer and distributed unknown input observer designs," *IEEE Transactions on Automatic Control*, vol. 69, no. 12, pp. 8868–8875, 2024.

[26] W. Zhang, Z. Wang, T. Raïssi, and Y. Shen, "Ellipsoid-based interval estimation for Lipschitz nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6802–6809, 2021.

[27] Y. Chen, J. Lam, and B. Zhang, "Estimation and synthesis of reachable set for switched linear systems," *Automatica*, vol. 63, pp. 122–132, 2016.

[28] W. Xiang, H.-D. Tran, and T. T. Johnson, "Reachable set estimation and control for switched linear systems with dwell-time restriction," in *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 7246–7251.

[29] S. Baldi and W. Xiang, "Reachable set estimation for switched linear systems with dwell-time switching," *Nonlinear Analysis: Hybrid Systems*, vol. 29, pp. 20–33, 2018.

[30] R. Zhao, Y. Wang, and Z. Zuo, "Reachable set estimation and synthesis of discrete-time switched systems," *International Journal of Robust and Nonlinear Control*, vol. 30, no. 18, pp. 8060–8073, 2020.

[31] L. I. Allerhand and U. Shaked, "Robust stability and stabilization of linear switched systems with dwell time," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 381–386, 2010.

[32] S. Yuan, L. Zhang, B. De Schutter, and S. Baldi, "A novel Lyapunov function for a non-weighted $\mathcal{L}_2$ gain of asynchronously switched linear systems," *Automatica*, vol. 87, pp. 310–317, 2018.

[33] C. Fan, J. Lam, X. Xie, and P. Li, "Stability and stabilization of almost periodic piecewise linear systems with dwell time uncertainty," *IEEE Transactions on Automatic Control*, vol. 68, no. 2, pp. 1130–1137, 2022.

[34] Z. Feng, X. Zhang, J. Lam, and C. Fan, "Estimation of reachable set for switched singular systems with time-varying delay and state jump," *Applied Mathematics and Computation*, vol. 456, p. 128132, 2023.

[35] A. Kundu and D. Chatterjee, "Stabilizing switching signals: A transition from point-wise to asymptotic conditions," *Systems & Control Letters*, vol. 106, pp. 16–23, 2017.

[36] C. Briat, "Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints," *Automatica*, vol. 49, no. 11, pp. 3449–3457, 2013.